

Is Stochastic Volatility relevant for Dynamic Portfolio Choice under Ambiguity?*

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Abstract

Literature on dynamic portfolio choice has been finding that volatility risk has low impact on portfolio decisions. For example, using long-run US data, Chacko and Viceira (2005) found that intertemporal hedging demand is empirically small even for highly risk-averse investors. We want to assess if this continues to be true in the presence of ambiguity about the stochastic investment opportunity set dynamics. Adopting robust control and perturbation theory techniques, we study the problem of a long-horizon investor with recursive preferences that faces ambiguity about the stochastic processes that generate the investment opportunity set. We find that ambiguity impacts optimal portfolio decisions, with the relevant channel being the return process. Ambiguity about the stochastic volatility process is only relevant as long as, through a specific correlation structure, it also induces ambiguity about the return process. Using long-run US data, we find that ambiguity about the return process may be empirically very relevant, much more than ambiguity about the volatility process. The optimal hedging demand required by investors for protection against adverse changes in volatility is still very low: investors are essentially focused in short term (myopic) risk-return characteristics of the risky asset, and uncertainty about volatility has low relevance for intertemporal portfolio decisions.

Keywords: Asset Allocation, Stochastic Volatility, Ambiguity, Perturbation Theory, Robust Control

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1 Introduction

We study optimal dynamic portfolio choice under a stochastic investment opportunity set, of a investor that is averse both to risk and ambiguity. We want to understand if and how ambiguity about the risky

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asset return and variance stochastic processes impact dynamic portfolio choices. More particularly, we want to assess if stochastic volatility continues to have low impact on portfolio decisions, as it has been found in the literature, in the presence of ambiguity about the stochastic investment opportunity set.¹

There is a large literature on portfolio choice (see, for e.g., Kogan and Uppal (2000) and Campbell and Viceira (2002) for a survey), but relatively few works study optimal dynamic portfolio choice with stochastic variance of the risky asset's return. Some examples are Kim and Omberg (1996) and Chacko and Viceira (2005) under incomplete markets, and Schroder and Skiadas (1999) under complete markets. Schroder and Skiadas (2003) contains a general closed-form solution for the consumption-portfolio problem, which includes the other models as special cases. Other set of papers consider multiple risky assets, as Liu (2007) and Buraschi et al. (2010). Potential adverse changes in the investment opportunity set are associated with the stochastic variance (and covariance in case of multiple risky assets) of the risky asset's return, which therefore represents a source of risk to investors. This implies, from Merton (1973), that stochastic variance is a key factor driving the investor's optimal intertemporal hedging demand.² However, in those papers there is only risk, and no ambiguity.

Ambiguity is uncertainty that cannot be represented by a single probability distribution. Risk, on the contrary, is uncertainty that is susceptible of being described by a probability distribution. This conceptual distinction, first explored by Knight (1921), has relevant implications for the behavior of economic agents, and therefore for economic theory in general. Ellsberg (1961) disclosed experimental evidence supporting the Knightian distinction between risk and ambiguity. This evidence became known as the Ellsberg paradox, and motivated a large body of empirical studies (surveyed, for e.g., in Camerer and Weber (1992) and Epstein and Schneider (2010)).

Notwithstanding this, the mainstream theory of choice under uncertainty in economics in the last 60 years ignored ambiguity, being based on the expected utility theory of von Neumann and Morgenstern (1944), where the probabilities of the possible states of nature are known, and on the subjective expected utility theory of Savage (1954), where although probabilities are not necessarily known, the choice behavior of an agent coincides with the maximization of expected utility according to some subjective probability beliefs.

However, gradually, ambiguity is being incorporated in decision theory. Two main approaches are being used: (i) the multiple priors (MP) approach, where the single probability measure of the standard expected and subjective utility models (precise beliefs) is replaced by a set of probabilities or priors (imprecise beliefs); (ii) the robust control (RC) approach, associated to an assumption of model uncertainty. The relationship between MP and RC approaches has been widely discussed in the literature, for e.g., in Hansen and Sargent (2001), Hansen et al. (2002), Epstein and Schneider (2003), and Maccheroni et al. (2006).

Ahn et al. (2011) have found empirical support for the relevance of studying the portfolio choice problem under ambiguity (about 2/3 of agents in their experience showed a positive degree of ambiguity aversion). Bossaerts et al. (2010) also conclude that ambiguity aversion can be observed in competitive

¹Throughout this paper by the term "volatility" we mean variance. Moreover, for mathematical convenience, we work with precision, the reciprocal of variance.

²Under the multivariate setting of Buraschi et al. (2010), with a stochastic variance-covariance matrix, there is an intertemporal hedging demand associated with stochastic variance and another associated with stochastic correlation between risky assets returns.

markets and that it matters for portfolio choices and for asset prices.

In studies of portfolio choice with ambiguity, Gollier (2006) and Garlappi et al. (2007) concluded that, by introducing ambiguity aversion in a static MP approach, the optimal demand for the risky asset decreases versus the standard mean-variance and Bayesian models.³ The same conclusion was reached in a dynamic MP setting (e.g. Chen et al. (2011)) and in a dynamic RC model (e.g. Maenhout (2004, 2006) and Xu et al. (2011)). The implications of ambiguity aversion for portfolio diversification have also been studied (Uppal and Wang (2003)). In all these works, with exception of Xu et al. (2011), the source of ambiguity is exclusively the expected risky asset's return or the risky asset's return process.

In this paper, we extend the model of Chacko and Viceira (2005) for optimal dynamic portfolio choice, by introducing ambiguity about the data generating process of the stochastic investment opportunity set. The motivation for this is twofold. On the one hand, it is provided by Chacko and Viceira themselves:

“An important caveat of our empirical analysis is that we have counterfactually assumed that investors observe volatility (or precision), and that they take as true parameters our empirical estimates of the joint process for returns and volatility. In practice, however, investors do not observe volatility, and they do not know the parameters of the process for volatility, or even the process itself.”

On the other hand, literature on dynamic portfolio choice with stochastic variance has been finding that variance risk has low impact on portfolio decisions (e.g. Chacko and Viceira (2005) and Liu (2007)). Recovering Knight (1921) uncertainty conceptualization, we want to understand if that evidence continues to be true if uncertainty is considered in a broader perspective, by taking into account an “ambiguity dimension” alongside the standard “risk dimension”.

It has been advocated in the literature (Cao et al. (2005), Garlappi et al. (2007) and Ui (2011)) that it is reasonable to assume that investors estimate the variance of the risky asset's return without ambiguity, and that it is preferable to assume ambiguity about expected returns. Reasons invoked for this are analytical tractability, empirical evidence on the predictability of the variance of stock returns (Bollerslev et al. (1992)), higher difficulty in estimating the expected returns versus expected variance (Merton (1980)) and higher costs associated with errors in estimating expected returns versus expected variance (Chopra and Ziemba (1993)).

Nevertheless, we introduce ambiguity also about the variance process of the risky asset's return because (i) there is no “a priori” reason to assume that investors are not ambiguous about it and (ii) we are able to find an asymptotic analytical solution and test it empirically.

In Faria et al. (2009), the setting of Chacko and Viceira (2005) is extended by considering a representative investor that is ambiguous about one specific parameter of the stochastic variance process (the expected value). It is adopted a MP approach and the conclusion is that ambiguity does not impact the instantaneous optimal portfolio rule. The ambiguity effect only exists, and is found to be empirically relevant, when the investor does not update continuously his portfolio. In Faria and Correia-da Silva (2010) we derive the optimal portfolio rule under a dynamic setting, with stochastic variance and ambiguity about its process. There it is assumed that the representative investor derives utility exclusively through terminal wealth, implying that the intertemporal consumption-savings de-

³Although the result in Gollier (2006) requires some restrictions on the set of priors and on the investor's attitude towards risk.

cision is ignored, and ambiguity is treated through a RC approach. Analytical expression for the optimal portfolio rule is derived, with ambiguity aversion having an additive impact to risk aversion. No empirical test is performed.

The closest paper to the present work is that of Xu et al. (2011), where preferences of the representative investor are given by the SDU function introduced by Duffie and Epstein (1992b) and ambiguity about the data generating process with stochastic variance is also considered and studied through a RC approach. Compared with Xu et al. (2011) contribution, the present paper brings three major novelties. The first results from the fact that we adopt a different RC methodology: the “constrained preferences” instead of the “multiplier preferences” approach (also applied in Faria and Correia-da Silva (2010)).⁴ Under “constraint preferences”, there is a constraint on the magnitude of the allowable perturbations from the benchmark model. Under “multiplier preferences”, preferences for robustness are constructed by penalizing deviations from the benchmark model, with higher deviations being more penalized than smaller ones. A relevant implication is that under “constrained preferences” ambiguity impacts the optimal portfolio through a non-uniform way in the state variables domain. In other words, the ambiguity aversion impact is more than simply an enhanced risk aversion level, bringing “first order” effects to optimal portfolio decisions. Moreover, in order to derive optimal policies under ambiguity, we make use of perturbation theory methods under RC settings, as for e.g. in Trojani and Vanini (2002, 2004). The rationale behind the perturbation (asymptotic) method is well described by Trojani and Vanini (2004): “... *formulate a general problem, find a particular relevant case that has a known solution, and use this as a starting point for computing the solution to nearby problems*”. In our case, as in Trojani and Vanini (2004), the asymptotic solution of the problem under ambiguity will hold for neighborhoods of a model with no ambiguity aversion.

The second difference towards Xu et al. (2011) is that we want to understand the relevant channels (return, variance or both) through which ambiguity impacts dynamic portfolio choice. For that, we study optimal dynamic portfolio choice when ambiguity is simultaneously about the return and volatility processes, as in Xu et al. (2011), and when it is exclusively about one of them.

The third difference versus Xu et al. (2011), is that we simulate our model using long-run US data, measuring the empirical dimension of the ambiguity impact on optimal portfolio decision. This is crucial as, ultimately, we are addressing the question if stochastic variance is relevant for portfolio decisions.

The main conclusions of this paper concern the impact on optimal dynamic policies from ambiguity about the data generating processes, both when ambiguity is simultaneously about the return and variance processes and when it is exclusively about one of them. In all scenarios, it is found that ambiguity does not impact the optimal consumption rule.

Conversely, ambiguity about the data generating processes reduces the optimal demand for the risky asset. The relevant channel is the return process, as when ambiguity is exclusively about the variance process there is no impact on the optimal portfolio rule: ambiguity about the variance process is only relevant as long as, through a specific correlation structure, it also induces ambiguity about the return process.

Using long-run US data, it is found that ambiguity about the stochastic processes driving the

⁴We adopt this terminology from Hansen and Sargent (2006).

investment opportunity set is empirically relevant for portfolio decisions. Our simulation suggests that ambiguity about the risky asset return process is empirically much more relevant than ambiguity about the stochastic variance process. We also conclude that even accounting for ambiguity about the stochastic variance process, the optimal hedging demand required by investors for protection against adverse changes in variance is still very low. Investors are essentially focused in short term risk-return characteristics of the risky asset, and stochastic volatility has low relevance for portfolio decisions: this has been found under settings where uncertainty is exclusively risk (for example in Chacko and Viceira (2005) and Liu (2007)) and we extend that conclusion for a setting where uncertainty also has an ambiguity dimension.

The paper is organized as follows. In section 2, we state the problem to be solved, in its base case scenario. In section 3, we present the analytical solution to that problem and the key results. In section 4 we analyze alternative scenarios for the sources of ambiguity, deriving analytical solutions and comparing its results with those of section 3. In section 5, simulation results are presented. In section 6, we conclude the paper with some remarks.

2 Consumption and Portfolio Choice Problem

In section 2.1, the investment opportunity set is described. In section 2.2, the preferences of the representative investor are presented. In section 2.3, the dynamic optimization problem to be solved is disclosed.

2.1 Investment Opportunity Set

It is assumed that all wealth is allocated between a riskless asset with price B_t and a risky asset with price S_t . The instantaneous return of the riskless asset is described by:

$$\frac{dB_t}{B_t} = r dt, \quad (1)$$

where r stands for the risk free interest rate.

The instantaneous return of the risky asset is given by:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{\frac{1}{y_t}} \left(\rho dW_y + \sqrt{1 - \rho^2} dW_\varepsilon \right), \quad (2)$$

where μ is the expected return of the risky asset and y_t is the instantaneous precision of the risky asset's return process (the instantaneous variance is $v_t = \frac{1}{y_t}$). W_ε and W_y are two independent standard Brownian motions.

The precision, y_t , follows a mean-reverting, square-root process as used by Cox et al. (1985):

$$dy_t = \kappa (\theta - y_t) dt + \sigma \sqrt{y_t} dW_y, \quad (3)$$

where the expected value of the precision is $E[y_t] = \theta$, the reversion parameter is $\kappa > 0$, and, thus, $Var[y_t] = \frac{\sigma^2 \theta}{2\kappa}$. To guarantee standard integrability conditions, it is assumed that $2\kappa\theta > \sigma^2$, as in Cox

et al. (1985).

Applying Itô's Lemma to (3), a mean-reverting, square-root process for proportional changes in variance is obtained:

$$\frac{dv_t}{v_t} = \kappa_v (\theta_v - v_t) dt - \sigma \sqrt{v_t} dW_y, \quad (4)$$

where $\theta_v = \left(\theta - \frac{\sigma^2}{\kappa}\right)^{-1}$ and $\kappa_v = \kappa \left(\theta - \frac{\sigma^2}{\kappa}\right) = \frac{\kappa}{\theta_v}$.

Taking expectations of the second-order Taylor expansion of v_t around θ , the approximate unconditional mean of instantaneous variance is:

$$E[v_t] \approx \frac{1}{\theta} + \frac{1}{2} \frac{\sigma^2}{\theta^2 \kappa} = \frac{1}{\theta} + \frac{Var(y_t)}{\theta^3}. \quad (5)$$

As the expected return of the risky asset, μ , is assumed to be constant, (5) is also the unconditional variance of the risky asset's return. Chacko and Viceira (2005) perform a Monte Carlo simulation that validates this statement and the accuracy of the approximation (5), concluding that this approximation understates the true variance by 0.27%.

It is implicit in (2) - (3) that shocks in precision (W_y) are correlated with shocks in the return of the risky asset, with correlation given by $\rho > 0$. From (4), this implies that the instantaneous correlation between proportional changes in variance and the risky asset's return is given by:

$$Corr_t \left(\frac{dv_t}{v_t}, \frac{dS_t}{S_t} \right) = -Corr_t \left(dy_t, \frac{dS_t}{S_t} \right) = -\rho dt. \quad (6)$$

This investment opportunity setting incorporates three of the main stylized facts about the variance of the return of risky assets: the mean reversion property, the "leverage effect" property (given by the negative correlation between return and its variance), and the fact that proportional changes in variance are higher when variance is high.

2.2 Investor Preferences

It is assumed that the representative investor is not totally sure about the stochastic processes (2)-(3) that generate the dynamic investment opportunity set. In other words, the uncertainty faced by the representative investor has two dimensions: risk and ambiguity.

Additionally, it is assumed that the preferences of the representative investor are described by the stochastic differential utility (SDU) function introduced by Duffie and Epstein (1992b) and applied to asset pricing theory in Duffie and Epstein (1992a). This is a continuous-time form of recursive utility, analogous to the discrete-time parametrization of Epstein and Zin (1989, 1991), that exhibits intertemporal consistency, admits Bellman's characterization of optimality, and separates risk aversion from elasticity of intertemporal substitution by not constraining to be reciprocals of one another (as in standard additive intertemporal utility function).

The utility process that defines the SDU function is represented by:

$$J = E_t \left[\int_t^\infty f(C_s, J_s) ds \right], \quad (7)$$

where C_s represents current consumption and J_s is the continuation utility for the consumption flow C , at time $t = s$, with infinite time horizon. In our setting, the function $f(C_s, J_s)$ is the normalized aggregator that generates J , defining a SDU function that represents the preferences introduced by Kreps and Porteus (1978). An explicit closed-form expression for that SDU utility function is not available.

We assume an elasticity of intertemporal substitution $\psi = 1$. This is due to (i) the fact that with $\psi = 1$ there is an exact solution of the Bellman equation that we will obtain and (ii) we conclude that main analytical and empirical results do not change under $\psi \neq 1$.⁵ With $\psi = 1$ the normalized aggregator $f(C, J)$ takes the form:

$$f(C, J) = \beta(1 - \gamma)J \left\{ \ln(C) - \frac{1}{1 - \gamma} \ln[(1 - \gamma)J] \right\}, \quad (8)$$

where $\gamma > 0$ is the coefficient of relative risk aversion and $\beta > 0$ is the rate of time preference. With $\gamma = 1$ it is obtained the standard power utility representation.

A remark regarding the preference for the timing of the resolution of risk. With the preference structure in Kreps and Porteus (1978), investors can have preference for early or late resolution of risk (as well as indifference), while the standard additive intertemporal utility function implies that investors are indifferent to the temporal resolution of risk. In the framework of Epstein and Zin (1989), the preference for temporal resolution of risk depends on the relationship between ψ and γ : if $\gamma > \frac{1}{\psi}$ ($<$, $=$) investors have preference for early (late, indifferent) resolution of risk. Our specification (7) from Duffie and Epstein (1992a), being the continuous-time limit of Epstein and Zin (1989), inherits this property. However, on the contrary of other streams of literature with Epstein and Zin (1989) preferences, for e.g. the “long-run risk” literature (from the seminal work of Bansal and Yaron (2004)), we do not restrict the investor to have preference for early resolution of risk. Two main reasons support this decision: (i) as our model evolves in a long-run setting, the possibility of the “cost” becoming higher than the “benefit” of planning advantages brought by the early resolution of risk (Arai (1997)) should not be excluded and (ii) there is evidence that investors may have preference for late resolution of risk (Epstein and Zin (1991)).

2.3 Dynamic Optimization Problem

Ambiguity about the investment opportunity set is studied with robust control (RC) techniques, firstly introduced in economics by Hansen and Sargent (1995). The representative investor considers a base model as his reference belief but, being ambiguous about it, considers a family of alternative models around it. Those alternative models are statistically difficult to distinguish from the benchmark model.

Under the RC approach, two main formulations have been used in the ambiguity related literature: the “constraint preferences” and the “multiplier preferences”. Under “constraint preferences” (e.g. in Hansen et al. (2006)), there is a constraint on the magnitude of the allowable perturbations from the benchmark model. Under “multiplier preferences” (e.g. in Maenhout (2006)), preferences for robustness are constructed by penalizing deviations from the benchmark model, with higher deviations

⁵In Chacko and Viceira (2005) it is found, for the general case of $\psi \neq 1$, an approximate solution for the Bellman equation we will obtain. That solution converges to the exact solution when $\psi = 1$.

being more penalized than smaller ones. Although both settings are related, through the Lagrange Multiplier Theorem (Hansen and Sargent (2006)), they end up being structurally very different.

One difference is that under the “constraint preferences” RC approach, the specification of the ambiguity aversion can be based on a rectangular set of priors, which guarantees a dynamically consistent preference ordering. In those cases, preferences can be represented by the recursive multiple priors utility (RMPU) specification (Chen and Epstein (2002) and Epstein and Schneider (2003)).⁶ Note however that, on the contrary of the RMPU setting, RC methodology is focused on formulating optimal dynamic rules where rectangularity is not a crucial property “*per se*” for the formulation of the set of priors of the model (see, for e.g., Trojani and Vanini (2004)).

Additionally, the “constraint preferences” approach allows ambiguity to have “first-order” effects on portfolio choice (as pointed out by Epstein and Schneider (2010)), that is, it enables the expansion of the range of qualitative behavior that can be rationalized versus the standard expected utility theory. This contrasts with the “multiplier preferences” RC approach, which is observationally equivalent to the expected utility theory: it enables reinterpretations of some results obtained under the expected utility theory, that can be quantitatively more appealing,⁷ but does not enlarge the spectrum of qualitative behavior that can be rationalized. In this sense, the “multiplier preferences” RC approach only brings “second-order” effects on portfolio choice, with the ambiguity aversion being in practice translated in an enhanced risk aversion level (as concluded for optimal dynamic portfolio choice in Maenhout (2004, 2006), Faria and Correia-da Silva (2010) and Xu et al. (2011)).

In this paper we adopt a “constraint preferences” RC approach. The ambiguous agent considers contaminations (alternative models), P^h , around his reference belief, P under which processes (2)-(3) evolve. The contaminations are assumed to be absolutely continuous with respect to P , and, therefore, are equivalently described by contaminating drift processes, h . In each of the alternative models, P^h , the Brownian motion becomes $W^h(t) = W(t) + \int_0^t h(s) ds$.⁸

An upper bound is imposed on the contaminating drift processes, h :

$$h^\top h \leq 2\eta, \tag{9}$$

where $\eta \geq 0$ is a parameter that can be interpreted as the level of ambiguity.

This bound (9) should be such that alternative models are statistically close to the “reference belief” model: otherwise the agent would easily distinguish among them and, consequently, would not face ambiguity. That is, η should be small. Moreover, the bound (9) constrains both the instantaneous time variation and the continuation value of the relative entropy between the reference belief, P , and any admissible contaminated belief, P^h . Trojani and Vanini (2004) explain that the set $\{h : h^\top h \in [0, 2\eta], \forall t \geq 0\}$ defines a rectangular set of priors because any process h (and therefore

⁶Rectangularity is the property that allows updating every prior under the recursive multiple priors utility through a Bayes rule. See Epstein and Schneider (2003) for details about this property. In Hansen and Sargent (2006) there is a comprehensive discussion of the dynamic consistency issue under the robust control approach.

⁷For example, as ambiguity aversion translates into a higher effective risk aversion, it is a contribution for the explanation of the equity premium puzzle.

⁸For tractability reasons, the analysis is restricted to the class of Markov-Girsanov kernels. The absolute continuity assumption between P and P^h guarantees the equivalence property between the probability measures and, consequently, that the Cameron-Martin-Girsanov theorem can be applied. Moreover, from this theorem and considering the diffusion family of models under consideration, all that a probability measure change implies is the change of the drift function of the stochastic processes.

any probability measure P^h) in this set corresponds to a selection of transition densities from t to $t + dt$, $t \geq 0$, such that $h^\top h \in [0, 2\eta]$.⁹

Considering the reference system (2)-(3), ambiguity is introduced through contaminating drift processes restricted to be $h = \begin{bmatrix} h^y & h^\varepsilon \end{bmatrix}^\top$. The class of admissible Markovian drift contaminations satisfying this restriction and the entropy bound (9) is denoted by \mathcal{H} .

Under an admissible contamination, P^h , the investment opportunity set is therefore described by:

$$\begin{cases} \frac{dS_t}{S_t} = \left(\mu + \sqrt{\frac{1}{y_t}} \rho h^y + \sqrt{\frac{1}{y_t}} \sqrt{(1 - \rho^2)} h^\varepsilon \right) dt + \sqrt{\frac{1}{y_t}} \rho dW_y + \sqrt{\frac{1}{y_t}} \sqrt{(1 - \rho^2)} dW_\varepsilon \\ dy_t = (\kappa(\theta - y_t) + \sigma \sqrt{y_t} h^y) dt + \sigma \sqrt{y_t} dW_y \end{cases} \quad (10)$$

Note that in the ‘‘contaminated’’ system (10) describing the investment opportunity set, the diffusion component continues to be driven by the same vector of independent Brownian motions as in (2)-(3).

With C_t , X_t and π_t representing the instantaneous consumption, wealth and fraction of wealth invested in the risky asset, wealth dynamics is given by:

$$dX_t = \pi_t X_t \frac{dS_t}{S_t} + (1 - \pi_t) X_t r dt - C_t dt.$$

Considering the risky asset return dynamics in (10), the intertemporal budget constraint faced by the ambiguous representative investor is given by:

$$dX_t = \left[\pi_t \left(\mu + \sqrt{\frac{1}{y_t}} \rho h^y + \sqrt{\frac{1}{y_t}} \sqrt{(1 - \rho^2)} h^\varepsilon - r \right) X_t + r X_t - C_t \right] dt + \pi_t \sqrt{\frac{1}{y_t}} X_t \left(\rho dW_y + \sqrt{1 - \rho^2} dW_\varepsilon \right). \quad (11)$$

The intertemporal optimization problem has a max-min structure: (i) Aversion towards ambiguity is introduced by assuming that, in the spirit of Gilboa and Schmeidler (1989), the representative investor chooses from the set of alternative models, the one that corresponds to the worst case scenario, i.e., the one associated with the lowest expected utility. This brings the min component; (ii) The investor has to choose the consumption function, $C : [t_0, +\infty[\rightarrow \mathbb{R}_+$, and the fraction of wealth invested in the risky asset throughout time, $\pi : [t_0, +\infty[\rightarrow \mathbb{R}$, which maximize his expected utility (7):

$$\sup_{\pi, C} \inf_{h \in \mathcal{H}} E_t^h \left[\int_t^\infty f(C_s, J_s) ds \right], \quad (12)$$

subject to the contaminated precision and wealth processes in (10) and (11), respectively.

⁹In Trojani and Vanini (2004), p.289, there is a detailed explanation supporting the rectangularity property of the present set of priors built under the constraint (9), and how this rectangular set of priors can be defined in the *k-ignorance* model of Chen and Epstein (2002).

The Bellman equation of this problem is:

$$0 = \sup_{\pi, C} \inf_{h^y, h^\varepsilon} \left\{ f(C, J) + \left[\pi_t \left(\mu + \sqrt{\frac{1}{y_t}} \rho h^y + \sqrt{\frac{1}{y_t}} \sqrt{(1-\rho^2)} h^\varepsilon - r \right) X_t + r X_t - C_t \right] J_X + \right. \\ \left. + [\kappa(\theta - y_t) + \sigma \sqrt{y_t} h^y] J_y + \frac{1}{2} \pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{Xy} \right\}, \quad (13)$$

where $f(C, J)$ is the normalized aggregator given in (8) and J_X , J_y , J_{XX} , J_{yy} and J_{Xy} are partial derivatives of value function J .

Solving for the optimal vector (h^y, h^ε) , i.e. for the optimal contamination, and placing that result into (13), the Bellman equation of the problem becomes (Appendix 7.1):

$$0 = \sup_{\pi, C} \left\{ f(C, J) + \pi_t (\mu - r) X_t J_X + r X_t J_X - C_t J_X + \kappa(\theta - y_t) J_y + \frac{1}{2} \pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} \right. \\ \left. + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{Xy} - \sqrt{2\eta \left(\sigma^2 y_t J_y^2 + 2\sigma \rho \pi_t X_t J_y J_X + \pi_t^2 \frac{1}{y_t} X_t^2 J_X^2 \right)} \right\}. \quad (14)$$

3 Optimal Consumption and Portfolio Rules

In general, obtaining closed form solutions under stochastic investment opportunity sets is difficult. This difficulty is further enhanced by the consideration of ambiguity. In this paper, we follow perturbation theory methodology under robust control (e.g. Trojani and Vanini (2002)) to describe the solution of the problem under study. As in Trojani and Vanini (2004), we extend the asymptotic methods in Kogan and Uppal (2000) from models based on standard expected utility to models with ambiguity considerations. This is allowed by the homotheticity nature of the robust control problem (10) - (11) - (12). This homothetic property will become clear by the structure of the value function that solves the problem and the corresponding optimal consumption-wealth and portfolio policies, which are wealth scale-invariant.¹⁰

The rationale behind the perturbation (asymptotic) method is well described by Trojani and Vanini (2004): “... *formulate a general problem, find a particular relevant case that has a known solution, and use this as a starting point for computing the solution to nearby problems*”. In our case, as in Trojani and Vanini (2004), the asymptotic solution of the problem under ambiguity will hold for neighborhoods of a model with no ambiguity aversion.

The first step is therefore to identify a set of parameters that parametrize the problem under study and specific parameters values for which the value function solution is known explicitly. In our case, there are two critical parameters for this step: the intertemporal elasticity of substitution ψ and the level of ambiguity η . In Chacko and Viceira (2005) the intertemporal optimization problem in section

¹⁰As explained in Trojani and Vanini (2002), studying non-homothetic robust control settings with perturbation methods is more difficult. Moreover, Maenhout (2004) points out some reasons to support the homotheticity assumption: “*Although economics exhibit growth, rates of return are stationary. Second, when the scale of the state variables matters, natural unit invariance of optimal decisions disappears and calibrations have to take this into account. Finally, homotheticity facilitates aggregation and the construction of a representative agent*”. As it is stated in Maenhout (2004), preserving this homotheticity property guarantees that “...*robustness will no longer wear off as wealth rises*”.

2 is studied but without ambiguity considerations ($\eta = 0$): it is found an exact solution of the problem for $\psi = 1$. Following the rational above described, we will therefore perturb the equilibrium of a benchmark economy with $\psi = 1$ and $\eta = 0$.

The value function J that solves (14), for $\psi = 1$ and $\eta = 0$, is given by:

$$J(X_t, y_t) = \exp\{g_0(y_t)\} \frac{X_t^{1-\gamma}}{1-\gamma}, \quad (15)$$

where $g_0(y_t) = Ay_t + B$, with A and B being given by

$$A = \frac{\gamma(1-\gamma) \left[\left(\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma} \right) \pm \sqrt{\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma} \right)^2 - \frac{\sigma^2(\mu-r)^2[\gamma(1-\rho^2)+\rho^2]}{\gamma^2(1-\gamma)}} \right]}{\sigma^2[\gamma(1-\rho^2)+\rho^2]}, \quad (16)$$

$$B = \frac{(1-\gamma)(\beta \ln \beta + r - \beta)}{\beta} + \frac{\kappa\theta}{\beta} A. \quad (17)$$

The sign of the square-root in A is “+” for $\gamma > 1$ and “-” for $\gamma \leq 1$ (Appendix 7.2).

In order to obtain the asymptotic expansions of the optimal policies of the problem under study, function g_0 in (15) is expanded in $(\sqrt{2\eta})$ to first order:

$$g(y_t, \sqrt{2\eta}) = g_0(y_t) + g_1(y_t) \sqrt{2\eta} + \mathcal{O}_g^2(\sqrt{2\eta}), \quad (18)$$

where $\mathcal{O}_g^2(\sqrt{2\eta})$ is a symbol representing terms of second order in $(\sqrt{2\eta})$. As it is immediate from (18), g_0 is the specification of function $g(y_t, \sqrt{2\eta})$ for the scenario when there is no ambiguity ($\eta = 0$). And, as it will be proved, g_0 is enough to characterize first order optimal policies under this optimization problem with ambiguity aversion.

Proposition 1 *Asymptotic optimal consumption and portfolio policies under ambiguity about the investment opportunity set dynamics (2)-(3), when $\psi = 1$ and $\gamma \geq \omega$, where $\omega = \frac{\sigma^2(\mu-r)^2 + 2\rho\sigma(\mu-r)(\beta+\kappa)}{(\beta+\kappa)^2 + \sigma^2(\mu-r)^2 + 2\rho\sigma(\mu-r)(\beta+\kappa)} < 1$, are:¹¹*

$$C_t = \beta X_t + \mathcal{O}_c^2(\sqrt{2\eta}), \quad (19)$$

$$\pi_t = \frac{1}{\gamma + \sqrt{\frac{2\eta}{G_0(y_t)}}} \left[(\mu-r)y_t + \left(1 - \frac{1}{1-\gamma} \sqrt{\frac{2\eta}{G_0(y_t)}} \right) \sigma \rho A y_t \right] + \mathcal{O}_\pi^2(\sqrt{2\eta}), \quad (20)$$

with $G_0(y_t) = \left[\left(\frac{(\mu-r)+\rho\sigma A}{\gamma} \right)^2 + \left(\frac{\sigma A}{1-\gamma} \right)^2 + \frac{2\rho\sigma A((\mu-r)+\rho\sigma A)}{(1-\gamma)\gamma} \right] y_t$ and A given by (16).

Proof. Appendix 7.3.

The first comment on the above Proposition, is that the domain of the problem solution depends

¹¹For $\gamma < \omega$, A in (16) is a complex number. From (15) this implies a complex value function J . As J is the optimized intertemporal utility, and utility functions are defined in the real space, the domain of the problem is restricted to values of parameters such that $\gamma \geq \omega$.

on the combination of the level of investor's risk aversion and on the characterization of the investment opportunity set dynamics (represented by ω). Note also that regarding the investor's preferences for the temporal resolution of risk, the domain of analysis $\gamma \geq \omega$ includes scenarios where the investor: has preference for late resolution of risk ($\omega \leq \gamma < 1$); has preference for early resolution of risk ($\gamma > 1$); is indifferent to that timing ($\gamma = 1$). Only scenarios where the investor has a strong preference for late resolution of risk ($\gamma < \omega$) are excluded.

From (19) and (20), it is immediate to conclude that when there is no ambiguity, $\eta = 0$, optimal consumption and portfolio rules are as follows:

$$C_t = \beta X_t, \quad (21)$$

$$\pi_t = \frac{1}{\gamma} (\mu - r) y_t + \frac{\sigma \rho}{\gamma} A y_t, \quad (22)$$

which are the results in Chacko and Viceira (2005).

Comparing (19) and (21) it is clear that the asymptotic optimal consumption rule under ambiguity continues to be to consume the same constant fraction β of wealth throughout time. In this setting, the ambiguity impact on consumption must be of order higher than one. Additionally, the fact of the optimal rule being a constant fraction of wealth means that the income and substitution effects on consumption that result from a change in the investment opportunity set are always exactly canceled out.

Conversely, from (20) and (22), it is immediate to concluded that ambiguity aversion has an impact on the optimal dynamic portfolio decision. And there are novelties regarding existing results in the literature. First, the optimal allocation to the risky asset is instantaneously impacted by investor's ambiguity (parameter η), which contrasts with results in Faria et al. (2009). Secondly, the optimal portfolio rule is a non-linear function of y_t , which differs from the linear relationship when there exists no ambiguity (e.g. Chacko and Viceira (2005)) or when ambiguity is considered through multiplier preferences RC techniques (e.g. Maenhout (2004, 2006); Faria and Correia-da Silva (2010) and Xu et al. (2011)).

The structure of the optimal portfolio rule under ambiguity (20) continues to be the sum of two well-known components from Merton (1973): (i) myopic portfolio demand, in this setting given by $\frac{(\mu - r)}{\gamma + \sqrt{\frac{2\eta}{G_0(y_t)}}} y_t$; and (ii) intertemporal hedging demand, given by $\frac{\left(1 - \frac{1}{1-\gamma} \sqrt{\frac{2\eta}{G_0(y_t)}}\right) \sigma \rho A}{\gamma + \sqrt{\frac{2\eta}{G_0(y_t)}}} y_t$. Comparing with the optimal portfolio rule without ambiguity (22), it continues to be true that the intertemporal hedging demand is zero (and therefore the portfolio myopic demand is optimal) when: the investor has unit coefficient relative risk aversion ($\gamma = 1$); investment opportunities are constant ($\sigma = 0$) or, being time-varying, it is not possible to use the risky asset to hedge against those changes ($\rho = 0$). However, the ratio between the myopic and intertemporal hedging demand components depends on y_t , on the contrary of what happens when there is no ambiguity ($\eta = 0$).

Additionally, without ambiguity an investor with $\gamma > 1$ has a negative intertemporal hedging demand, and the opposite when $\omega \leq \gamma < 1$, which is consistent with the findings in Chacko and Viceira (2005). When risk aversion is low ($\omega \leq \gamma < 1$), the investor is ready to support a worse performance when precision is low for extra performance when precision is high (recall that $\rho > 0$).

An investor with high risk-aversion ($\gamma > 1$) is not willing to accept this trade-off. The introduction of ambiguity makes this relation not so trivial: for low risk averse investors ($\gamma < 1$) but sufficiently highly ambiguous (high η) the intertemporal hedging also becomes negative.¹²

Note that the myopic and intertemporal hedging demand components under ambiguity aversion ($\pi_t^M(\eta)$ and $\pi_t^H(\eta)$) can be rewritten:

$$\pi_t^M(\eta) = \pi_t^M(0) \frac{\gamma}{\gamma + \sqrt{\frac{2\eta}{G_0(y_t)}}}, \quad (23)$$

$$\pi_t^H(\eta) = \pi_t^H(0) \frac{\gamma \left(1 - \frac{1}{1-\gamma} \sqrt{\frac{2\eta}{G_0(y_t)}}\right)}{\gamma + \sqrt{\frac{2\eta}{G_0(y_t)}}}, \quad (24)$$

where $\pi_t^M(0)$ and $\pi_t^H(0)$ represent the myopic and intertemporal hedging demand components without ambiguity aversion. It is immediate to conclude that ratios $\pi_t^M(\eta)/\pi_t^M(0)$ and $\pi_t^H(\eta)/\pi_t^H(0)$ are y_t dependent functions (due to $G_0(y_t)$), and that ambiguity affects both components through a non-uniform way in the state variable y_t domain.

It is easy to demonstrate that $G_0(y_t) > 0$ which, from (23) and (24), implies that the reduction in the optimal risky asset demand is a positive function of the level of ambiguity of the representative investor (higher η). The result that ambiguity aversion reduces the demand for the risky asset is the standard result within the still recent literature on portfolio choice under ambiguity. We extend this result to a setting where stochastic precision is one of the sources of ambiguity, and ambiguity is treated through a ‘‘constraint preferences’’ RC setting.

4 Alternative Scenarios

In section 4.1, optimal asymptotic consumption and portfolio rules are derived when there is ambiguity exclusively about the risky asset return stochastic process. In section 4.2, the same is done for a scenario with ambiguity only about the precision stochastic process.

4.1 Ambiguity exclusively about the return stochastic process

Considering the investment opportunity set described in section 2.1, the particular case when there exists ambiguity only about the risky asset return stochastic process is represented by the vector $h = \begin{bmatrix} 0 & h^\varepsilon \end{bmatrix}^\top$, i.e., $h^y = 0$. From (13), the corresponding Bellman equation is given by:

$$0 = \sup_{\pi, C} \inf_{h^\varepsilon} \left\{ f(C, J) + \left[\pi_t \left(\mu + \sqrt{\frac{1}{y_t}} \sqrt{(1-\rho^2)h^\varepsilon} - r \right) X_t + rX_t - C_t \right] J_X + \right. \\ \left. + \kappa(\theta - y_t) J_y + \frac{1}{2} \pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{Xy} \right\}. \quad (25)$$

¹²For this to happen under the $\omega \leq \gamma < 1$ domain, it is necessary that $\gamma > 1 - \sqrt{\frac{2\eta}{G_0(y_t)}}$.

From (9), it is immediate to conclude that with $h = \begin{bmatrix} 0 & h^\varepsilon \end{bmatrix}^\top$ the optimal contamination h^ε is a corner solution ($-\sqrt{2\eta}$ or $\sqrt{2\eta}$). Considering that (i) h^ε is the control variable for the *inf* dimension of the optimization problem and (ii) the term related with h^ε in (25) is $\pi_t \sqrt{\frac{1}{y_t}} \sqrt{(1-\rho^2)} X_t J_X h^\varepsilon$ and that $\pi_t \sqrt{\frac{1}{y_t}} \sqrt{(1-\rho^2)} X_t J_X > 0$, then the optimal contamination is $h^\varepsilon = -\sqrt{2\eta}$. Introducing this result into (25), the Bellman equation of the problem when $h = \begin{bmatrix} 0 & h^\varepsilon \end{bmatrix}^\top$ becomes:

$$0 = \sup_{\pi, C} \left\{ f(C, J) + \left[\pi_t \left(\mu - \sqrt{\frac{1}{y_t}} \sqrt{(1-\rho^2)} \sqrt{2\eta} - r \right) X_t + r X_t - C_t \right] J_X + \kappa (\theta - y_t) J_y + \frac{1}{2} \pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{Xy} \right\}. \quad (26)$$

Following the same rational as in section 3, optimal portfolio and consumption rules are deduced for the particular case with ambiguity only about the return stochastic process.

Proposition 2 *Asymptotic optimal consumption and portfolio policies under ambiguity about the risky asset return dynamics (2), when $\psi = 1$ and $\gamma \geq \omega$, where $\omega = \frac{\sigma^2(\mu-r)^2 + 2\rho\sigma(\mu-r)(\beta+\kappa)}{(\beta+\kappa)^2 + \sigma^2(\mu-r)^2 + 2\rho\sigma(\mu-r)(\beta+\kappa)} < 1$, are:*

$$C_t = \beta X_t + \mathcal{O}_c^2(\sqrt{2\eta}), \quad (27)$$

$$\pi_t = \frac{1}{\gamma} (\mu - r) y_t - \frac{\sqrt{1-\rho^2} \sqrt{2\eta}}{\gamma} \sqrt{y_t} + \frac{\sigma\rho}{\gamma} A y_t + \mathcal{O}_\pi^2(\sqrt{2\eta}), \quad (28)$$

with A given by (16).

Proof. Appendix 7.3.

The main conclusions from Proposition 1 are extendable to Proposition 2. The main difference versus the general case (Proposition 1) is that the ambiguity adjusted optimal portfolio rule (28) only differs from the one when there is no ambiguity (22) at the myopic demand level, as the intertemporal hedging demand is the same, given by $\frac{\sigma\rho}{\gamma} A y_t$. This is expectable because the intertemporal hedging demand is driven by the stochastic variance dynamics and, in this particular case, nothing changes as ambiguity is only about the return process.

4.2 Ambiguity exclusively about the precision stochastic process

Under the investment opportunity set described in section 2.1, it is not possible to consider the particular case when there exists ambiguity only about the precision stochastic process. For that, we consider an alternative setting of (2)-(3) given by:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{\frac{1}{y_t}} dW_S, \quad (29)$$

$$dy_t = \kappa (\theta - y_t) dt + \sigma \sqrt{y_t} \left(\rho dW_S + \sqrt{1-\rho^2} dW_\varepsilon \right), \quad (30)$$

which keeps unchanged the properties of correlation between shocks in the return and precision processes (versus the setting (2)-(3)).

Considering now the reference system (29)-(30), ambiguity is again introduced through Markovian contaminating drift processes restricted to be $h = \begin{bmatrix} h^s & h^\varepsilon \end{bmatrix}^\top$ and satisfying the entropy bound (9), as in section 2.3.

Under an admissible contamination, P^h , the investment opportunity set is described by:

$$\begin{cases} \frac{dS_t}{S_t} = \left(\mu + \sqrt{\frac{1}{y_t}} h^s \right) dt + \sqrt{\frac{1}{y_t}} dW_S, \\ dy_t = \left[\kappa (\theta - y_t) + \sigma \sqrt{y_t} \rho h^s + \sigma \sqrt{y_t (1 - \rho^2)} h^\varepsilon \right] dt + \sigma \sqrt{y_t} \left(\rho dW_S + \sqrt{1 - \rho^2} dW_\varepsilon \right), \end{cases} \quad (31)$$

and the corresponding intertemporal budget constraint faced by the ambiguous representative investor is given by:

$$dX_t = \left[\pi_t \left(\mu + \sqrt{\frac{1}{y_t}} h^s - r \right) X_t + r X_t - C_t \right] dt + \pi_t \sqrt{\frac{1}{y_t}} X_t dW_S. \quad (32)$$

The Bellman equation of the optimization problem under this setting is:

$$\begin{aligned} 0 = & \sup_{\pi, C} \inf_{h^s, h^\varepsilon} \left\{ f(C, J) + \left[\pi_t \left(\mu + \sqrt{\frac{1}{y_t}} h^s - r \right) X_t + r X_t - C_t \right] J_X + \frac{1}{2} \pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} + \right. \\ & \left. + \left[\kappa (\theta - y_t) + \sigma \sqrt{y_t} \rho h^s + \sigma \sqrt{y_t (1 - \rho^2)} h^\varepsilon \right] J_y + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{Xy} \right\}. \end{aligned} \quad (33)$$

Solving for the optimal vector (h^s, h^ε) and placing that result into (33), the Bellman equation of the problem becomes (Appendix 7.4):

$$\begin{aligned} 0 = & \sup_{\pi, C} \left\{ f(C, J) + \pi_t (\mu - r) X_t J_X + r X_t J_X - C_t J_X + \kappa (\theta - y_t) J_y + \frac{1}{2} \pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} \right. \\ & \left. + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{Xy} - \sqrt{2\eta \left(\sigma^2 y_t J_y^2 + 2\sigma \rho \pi_t X_t J_y J_X + \pi_t^2 \frac{1}{y_t} X_t^2 J_X^2 \right)} \right\}, \end{aligned} \quad (34)$$

which is equal to (14). Consequently, optimal portfolio and consumption rules under the setting (29)-(30) are exactly those given in Proposition 1.

Consider now the particular case when there exists ambiguity only about the precision stochastic process, i.e. $h = \begin{bmatrix} 0 & h^\varepsilon \end{bmatrix}^\top$. For the same reasons as in section 4.1, the optimal contamination h^ε is a corner solution $(-\sqrt{2\eta}$ or $\sqrt{2\eta})$. In (33), the only element where h^ε enters is $\sigma \sqrt{y_t (1 - \rho^2)} J_y h^\varepsilon$, implying that, independently of what is the optimal h^ε , either $-\sqrt{2\eta}$ or $\sqrt{2\eta}$, the optimal consumption and portfolio rules are given by:

Proposition 3 *Asymptotic optimal consumption and portfolio policies under ambiguity about the precision stochastic process (30), when $\psi = 1$ and $\gamma \geq \omega$, where $\omega = \frac{\sigma^2 (\mu - r)^2 + 2\rho \sigma (\mu - r) (\beta + \kappa)}{(\beta + \kappa)^2 + \sigma^2 (\mu - r)^2 + 2\rho \sigma (\mu - r) (\beta + \kappa)} <$*

1, are:

$$C_t = \beta X_t + \mathcal{O}_c^2(\sqrt{2\eta}), \quad (35)$$

$$\pi_t = \frac{1}{\gamma}(\mu - r)y_t + \frac{\sigma\rho}{\gamma}Ay_t + \mathcal{O}_\pi^2(\sqrt{2\eta}), \quad (36)$$

with A given by (16).

The main conclusion from Proposition 3 is that when there exists ambiguity only about the precision stochastic process, there is no impact from that ambiguity on optimal consumption and portfolio rules, at least until a first order approximation with respect to the level of ambiguity.

Overall, optimal portfolio rules deduced for the scenarios where there exists ambiguity about the dynamics of both the return and its variance (Proposition 1), only about the return dynamics (Proposition 2) and only about variance dynamics (Proposition 3), show that from a theoretical point of view, the relevant channel through which ambiguity impacts portfolio decisions is the return of the risky asset. Ambiguity about the stochastic variance process is only relevant as long as, through a specific correlation structure, it also induces ambiguity about the return stochastic process: this happens under the setting (2)-(3), but not under the setting (29)-(30). In the next section we evaluate the empirical relevance of those findings.

5 Simulation

Chacko and Viceira (2005) found that, calibrating their model to long-run US data, the optimal intertemporal hedging demand is empirically small. The same conclusion is reached in Liu (2007). This suggests that the “risk dimension” of stochastic variance is empirically not very relevant to the dynamic optimal portfolio decision. However, Chacko and Viceira (2005) in their concluding remarks, acknowledged that an important caveat of their analysis is that they have counterfactually assumed that investors observe variance and take as true the empirical estimates of the parameters of the variance process.

Following this lead, we have set up a model to account for ambiguity about the same stochastic investment opportunity set of Chacko and Viceira (2005). As a result, the “myopic demand” and the “intertemporal hedging demand” become “ambiguity” adjusted.

Using the estimation of Chacko and Viceira (2005), our simulation suggests that the ambiguity impact on the allocation to the risky asset has a relevant empirical dimension. However, this effect is essentially due to ambiguity about the return process, with ambiguity about the stochastic variance process being empirically very low.

The reference parameter values used in the simulation are those estimated in Chacko and Viceira (2005) based on monthly excess stock returns on the CRSP value-weighted portfolio over the T-Bill

rate from January 1926 through December 2000:

$$\begin{aligned}
 \mu - r &= 0.0811, \\
 \kappa &= 0.3374, \\
 \theta &= 27.9345, \\
 \sigma &= 0.6503, \\
 \rho &= 0.5241, \\
 r &= 0.015, \\
 \beta &= 0.06.
 \end{aligned}
 \tag{37}$$

From (5), the expected standard deviation of returns is 19.1314%.

Implications on the optimal allocation to the risky asset from the consideration of ambiguity (Proposition 1) are exemplified in Table 1. The first column presents results for the scenario without ambiguity. The other three columns represent scenarios with three arbitrary levels of ambiguity: $\eta = 0.005, 0.01, 0.02$. Recall that alternative models have to be statistically close so that the investor is ambiguous about the reference model. This implies small η values. In Trojani and Vanini (2004) two arbitrary values for η are used (0.005, 0.01) while the value implied by all calibrations in Gagliardini et al. (2009) is lower than 0.0136.¹³

Simulations are run for different levels of risk aversion ($\gamma = 0.75, 1, 2, 4, 20, 40$) with unit elasticity of intertemporal consumption ($\psi = 1$). In panel A it is showed the mean allocation to the risky asset (percentage). In panel B, the intertemporal hedging demand is shown as a percentage of the myopic demand. In panels C and D, the ambiguity effect is explicitly calculated as a percentage of total risky asset demand and of myopic demand.

¹³Those η values can be taken as a reference without introducing any kind of bias in our analysis. This is because, as explained in section 2.3, the diffusion dimension of “contaminated” processes is unchanged vs. non-contaminated processes (only the drift functions are affected).

Table 1: Ambiguity impact on optimal risky asset demand (Proposition 1).

	η			
	0	0.005	0.01	0.02
A - Mean allocation to risky asset (%)				
γ				
0.75	305,66	245,80	227,16	204,98
1.00	226,55	180,98	166,91	150,24
2.00	111,37	89,59	82,80	74,73
4.00	55,24	44,43	41,07	37,07
20.00	10,98	8,83	8,16	7,37
40.00	5,48	4,41	4,08	3,68
B - Ratio of hedging demand over myopic demand (%)				
γ				
0.75	1,19	0,36	0,01	-0,48
1.00	0,00	-0,84	-1,19	-1,88
2.00	-1,88	-2,46	-2,78	-3,24
4.00	-2,47	-3,24	-3,56	-4,01
20.00	-3,09	-3,85	-4,16	-4,61
40.00	-3,17	-3,92	-4,24	-4,88
C - Reduction in allocation to risky asset due to ambiguity (%)				
γ				
0.75	0,00	-19,58	-25,68	-32,94
1.00	0,00	-19,57	-25,67	-32,92
2.00	0,00	-19,56	-25,65	-32,90
4.00	0,00	-19,56	-25,65	-32,89
20.00	0,00	-19,55	-25,64	-32,88
40.00	0,00	-19,55	-25,64	-32,88
D - Reduction of myopic demand due to ambiguity (%)				
γ				
0.75	0,00	-18,92	-24,81	-31,81
1.00	0,00	-18,92	-24,81	-31,81
2.00	0,00	-18,92	-24,81	-31,81
4.00	0,00	-18,92	-24,81	-31,81
20.00	0,00	-18,92	-24,81	-31,81
40.00	0,00	-18,92	-24,81	-31,81

Note 1 - Panel A: $E[\pi_t(y_t)] \times 100$, with $E(y_t) = \theta$; **Panel B:** $\frac{(1 - \frac{1}{1-\gamma}) \sqrt{\frac{2\eta}{G_0(E(y_t))}} \sigma \rho A}{(\mu - r)} \times 100$, with $E(y_t) = \theta$; **Panel C:**

$$\left[\frac{\binom{20}{22} - 1 \right] \times 100, \text{ with } E(y_t) = \theta; \text{ Panel D: } \left[\frac{\gamma}{\gamma + \sqrt{\frac{2\eta}{G_0(E(y_t))}}} - 1 \right] \times 100, \text{ with } E(y_t) = \theta;$$

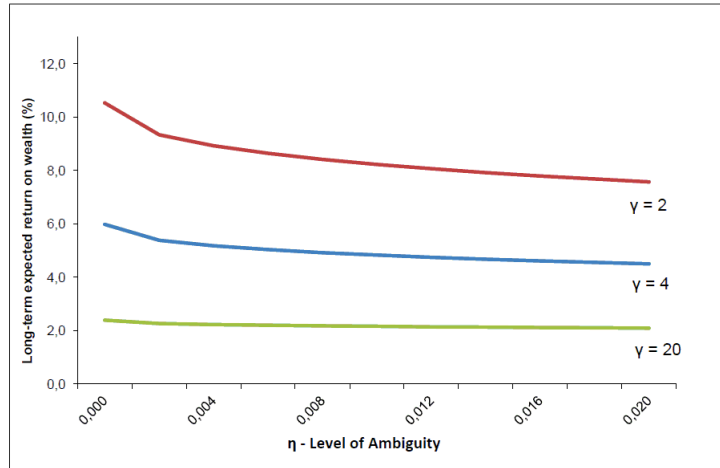
Note 2 - $\omega = 0.14 < 0.75$: domain of Proposition 1 is guaranteed.

Results presented in Table 1 are consistent with comments in section 3 and show that ambiguity is empirically relevant: even for a low level of ambiguity (second column in Table 1), ambiguity implies a 20% decrease of the mean optimal demand of the risky asset (panel C). Consider for example a risk-averse investor, with $\gamma = 2$, that is ambiguity neutral. His mean optimal allocation to the risky asset corresponds to 111.4% of his wealth. If this investor becomes ambiguous, for example with ambiguity level given by $\eta = 0.01$, his mean optimal allocation to the risky asset declines to 82,8% of his wealth. Those findings can be a contribute for the explanation of the so called “flight to quality” stylized fact in financial markets: when investors for some reason become more “nervous” and uncertain about market conditions, they reduce exposure to risky assets and “fly” towards less risky or riskless assets.

Panel A in Table 1 shows that the demand for the risky asset is decreasing with the risk aversion, γ , and with the level of ambiguity, η . Considering that the long-term expected return on wealth is measured by $(\pi_\theta(\mu - r) + r)$, results that it is a decreasing function of both risk aversion and the level

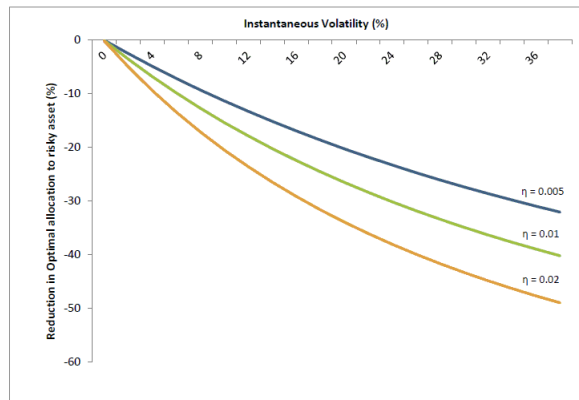
of ambiguity.

Figure 1: Long-term expected return on wealth as a function of ambiguity and risk aversion.



Moreover, the higher the level of ambiguity and the level of instantaneous volatility (inverse of precision) the higher is the impact from ambiguity. This is represented in Figure 2, where it is also evident that this ambiguity effect through the relevant support of precision or, equivalently, of variance has a non-linear nature (concave and convex, respectively).

Figure 2: Ambiguity effect as a function of instantaneous volatility



Note 1 - Volatility understood as variance v_t , computed from the level of precision y_t using (5);

Note 2 - Simulation with $\gamma = 4$.

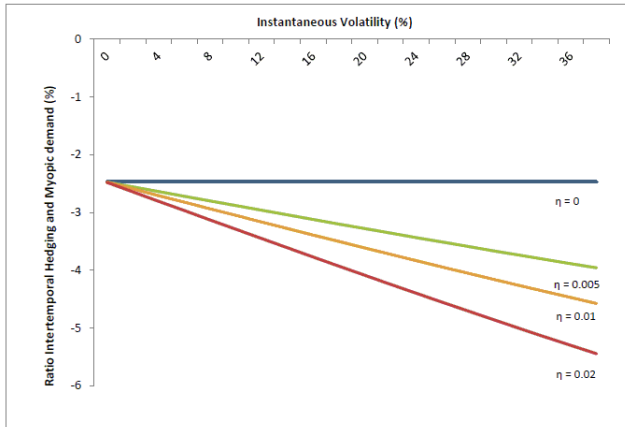
Those findings suggest that the intensity, i.e. the speed and depth, of the asset reallocation implied in the “flight to quality” phenomena should increase with the level of ambiguity and of instantaneous volatility. This is intuitive: in an anxious market environment as the one following Lehman Brothers collapse in September 2008 (which Blanchard (2009) suggestively named as “Knight time”), during which the VIX index¹⁴ reached its historical maximum of 80.86% (20th November 2008), the speed

¹⁴The VIX Index from CBOE is probably the most used volatility index, both in the literature and in the industry. It measures the one-month implied volatility in the S&P 500 Index option prices. For full details on the VIX Index

and volumes of risky asset “sell-off” trades were much higher than in stable market conditions.

Additionally, as pointed out in section 3, the ratio between the intertemporal hedging demand and myopic demand components becomes y_t - state dependent when ambiguity is considered. This has an intuitive interpretation: the higher the level of instantaneous volatility, everything else constant, the more the investor is concerned about the impact from volatility changes in his intertemporal utility and, being ambiguous about the process that drives volatility, the higher is the optimal hedging demand. This is graphically highlighted in Figure 3, where it is also clear that the dimension of this adjustment is empirically small:

Figure 3: Hedging Demand vs Myopic Demand as a function of instantaneous volatility



Note 1 - Volatility understood as variance v_t , computed from the level of precision y_t using (5);

Note 2 - Simulation with $\gamma = 4$.

In Panel B it is reported estimates of the intertemporal hedging demand, measured as a ratio of myopic demand. Again, results show that ambiguity reinforces the effect of risk aversion: the higher the ambiguity and risk aversion the higher the relative dimension of the intertemporal hedging demand. However, the intertemporal hedging demand is always small - even for a very high risk and ambiguity averse investor ($\gamma = 40, \eta = 0.02$). The novelty with ambiguity is that for low risk averse investors but with high ambiguity the intertemporal hedging demand becomes negative (e.g. $\gamma = 0.75, \eta = 0.02$). This confirms the predictions highlighted in section 3. With no ambiguity or moderate levels of ambiguity the intertemporal hedging demand is positive when $\gamma < 1$ and negative when $\gamma \geq 1$.

The fact of the intertemporal hedging demand component being empirically small, even for higher levels of ambiguity, means that ambiguity impacts optimal portfolio decision essentially through the myopic component, which is confirmed by results disclosed in Panels C and D. This has a clear economic meaning and provides an answer to the research question addressed in this paper: even accounting for ambiguity about the stochastic volatility process, it is found that the optimal hedging demand required by investors for protection against adverse changes in volatility is still very low. Investors are essentially focused in short term risk-return characteristics of the risky asset (myopic dimension), and stochastic volatility has low relevance for optimal intertemporal portfolio decisions: this has been

construction methodology please see <http://www.cboe.com/micro/vix/>.

found under settings where uncertainty is exclusively risk (for example in Chacko and Viceira (2005) and Liu (2007)) and we extend the conclusion for a setting where uncertainty also has an ambiguity dimension.

Moreover, recalling conclusions from section 4.2, this setting (Table 1) is the only one where ambiguity about stochastic volatility process impacts optimal portfolio decisions.

At last, as a “cross-check” test, the scenario with ambiguity exclusively about the risky asset return process (Proposition 2) was simulated, confirming the empirical relevance of such level of ambiguity:

Table 2: Ambiguity impact on optimal risky asset demand (Proposition 2).

	η			
	0	0.005	0.01	0.02
	A - Mean allocation to risky asset (%)			
γ				
0.75	305,66	245,64	220,78	185,62
1.00	226,55	181,54	162,89	136,52
2.00	111,37	88,87	79,55	66,36
4.00	55,24	43,98	39,32	32,73
20.00	10,98	8,73	7,79	6,48
40.00	5,48	4,36	3,89	3,23
	B - Reduction in allocation to risky asset due to ambiguity (%)			
γ				
0.75	0,00	-19,64	-27,77	-39,27
1.00	0,00	-19,87	-28,10	-39,74
2.00	0,00	-20,21	-28,58	-40,42
4.00	0,00	-20,37	-28,81	-40,75
20.00	0,00	-20,50	-28,99	-41,01
40.00	0,00	-20,52	-29,02	-41,04

Note 1 - Panel A: $E[\pi_t(y_t)] \times 100$, with $E(y_t) = \theta$; **Panel B:** $\left[\frac{\binom{28}{22}}{\binom{28}{22}} - 1\right] \times 100$, with $E(y_t) = \theta$;

Note 2 - $\omega = 0.14 < 0.75$: domain of Proposition 2 is guaranteed.

Comparing results in Panel B of Table 2 with those in Panel C of Table 1, leads to a curious finding: although the aggregate level of ambiguity is the same in both settings ($\eta = 0, 0.005, 0.01, 0.02$), as in Table 2 ambiguity is concentrated in the relevant channel (returns) its impact on the optimal demand for the risky asset is higher.

Our conclusion that uncertainty about stochastic volatility has low impact on portfolio decisions may change significantly if a multivariate stochastic variance setting (multiple risky assets) is considered, as for example in Buraschi et al. (2010). The authors show that joint features of volatility and correlation dynamics play an important role in optimal portfolios. As an example, Buraschi et al. (2010) estimate that, under a univariate stochastic volatility model, total hedging demand for S&P 500 futures of investors with $\gamma = 8$ and investment horizon of 10 years is 4.8% of the myopic demand. This is consistent with our results in Table 1. However, for a three risky asset model, the estimated total hedging demand for S&P500 futures jumps to 28% of myopic demand, with 11% and 17% of volatility and correlation hedging demand respectively.

Interestingly, Buraschi et al. (2010) also find that the optimal hedging demand against correlation risk typically dominates hedging against volatility risk: this is, at least partially, explained by the higher persistence of correlation risk versus that of volatility risk. In a study about the relation between correlation risk and the cross-section of hedge fund returns, Buraschi et al. (2012) find evidence

that correlation risk is the most significant risk factor for the explanation of hedge fund returns. Those findings suggest that correlation, more than volatility, is the relevant uncertainty factor to be “controlled” and therefore commanding investors intertemporal hedging demand.

6 Concluding Remarks

We study optimal dynamic consumption and portfolio choice with stochastic variance, by introducing ambiguity about the data generating processes driving the stochastic investment opportunity set.

Long-horizon investors with recursive preferences, as defined by Duffie and Epstein (1992b) with Kreps and Porteus’ (1978) specification, have two assets to invest in, a risk-free asset and a risky asset. The investor considers a reference model for the data generating processes but, not being totally sure about it, takes into account a set of statistically close models (with the relative entropy between models being bounded). Ambiguity aversion in the spirit of Gilboa and Schmeidler’s (1989) implies that investor will consider the worst possible alternative model, i.e., the one associated with the lowest expected utility. Optimal dynamic policies under ambiguity are deduced by making use of perturbation theory techniques for robust control problems.

The main conclusions of this paper concern the impact on optimal dynamic policies from ambiguity about the data generating processes, both when ambiguity is simultaneously about the return and volatility processes and when it is exclusively about one of them. In all scenarios, the optimal consumption policy is to consume a constant fraction of wealth. It is found that ambiguity does not impact the optimal consumption-wealth ratio, at least until a first order approximation with respect to the level of ambiguity.

Conversely, ambiguity about the data generating processes reduces the optimal demand for the risky asset, with that effect being non-uniform in the variance domain. The same happens when there is ambiguity only about the risky asset return. When ambiguity is exclusively about the stochastic variance process it is found that there is no impact on the optimal portfolio rule. The conclusion is that ambiguity about the stochastic variance process is only relevant as long as, through a specific correlation structure, it also induces ambiguity about the return stochastic process.

Making use of long-run US data, we measure the empirical dimensions of those effects. The first conclusion is that ambiguity about the stochastic processes driving the investment opportunity set is empirically relevant for portfolio decisions. This can be a contribute for the explanation of the “fly to quality” stylized fact in financial markets. Our simulation suggests that this highly relevant ambiguity effect on the risky asset demand acts mainly through the myopic component. The first implication is the confirmation that, under our setting and simulation, ambiguity about the risky asset return process is empirically much more relevant than ambiguity about stochastic volatility process. The second implication is that, even accounting for ambiguity about the stochastic volatility process, it is found that the optimal hedging demand required by investors for protection against adverse changes in volatility is still very low. Investors are essentially focused in short term risk-return characteristics of the risky asset (myopic dimension), and stochastic volatility has low relevance for intertemporal portfolio decisions: this has been found under settings were uncertainty is exclusively risk (e.g. in Chacko and Viceira (2005) and Liu (2007)) and we extend that conclusion for a setting

where uncertainty also has an ambiguity dimension.

Our conclusion that uncertainty about stochastic volatility has a low impact on portfolio decisions may change significantly if a multivariate stochastic variance setting (multiple risky assets) is considered, as Buraschi et al. (2010) shows: the authors find that hedging demands are typically four to five times higher than those of models with constant correlations or single-factor stochastic volatility. Moreover, Buraschi et al. (2010) also find that correlation risk hedging demand is typically higher than volatility risk demand suggesting that correlation, more than volatility, is the crucial uncertainty factor to hedge in an intertemporal optimization portfolio problem.

An interesting research topic for future work is therefore the consideration of ambiguity about stochastic variance-covariance dynamics in a multivariate model for optimal intertemporal portfolio choice.

7 Appendices

7.1 Bellman Equation (14)

Starting to define Λ as the diffusion matrix of state variables y_t and X_t processes in (10) and (11):

$$\Lambda = \begin{bmatrix} \sigma\sqrt{y_t} & 0 \\ \rho\pi_t\sqrt{\frac{1}{y_t}}X_t & \pi_t\sqrt{\frac{1}{y_t}}X_t\sqrt{1-\rho^2} \end{bmatrix}. \quad (38)$$

The minimization of (13) with respect to the vector h gives (see, for e.g., Anderson et al. (1998) p. 22):

$$\begin{bmatrix} h^y \\ h^\varepsilon \end{bmatrix} = -\frac{\sqrt{2\eta}}{\sqrt{\begin{bmatrix} J_y & J_X \end{bmatrix} \Lambda \Lambda^\top \begin{bmatrix} J_y & J_X \end{bmatrix}^\top}} \Lambda^\top \begin{bmatrix} J_y \\ J_X \end{bmatrix}. \quad (39)$$

Replacing (38) in (39), the vector of optimal contaminating drifts is obtained and given by:

$$\begin{bmatrix} h^y \\ h^\varepsilon \end{bmatrix} = -\frac{\sqrt{2\eta}}{\sqrt{J_y^2\sigma^2y_t + 2J_yJ_X\sigma\rho\pi_tX_t + J_X^2\pi_t^2\frac{1}{y_t}X_t^2}} \begin{bmatrix} \sigma\sqrt{y_t}J_y + \rho\pi_t\sqrt{\frac{1}{y_t}}X_tJ_X \\ \pi_t\sqrt{\frac{1}{y_t}}X_t\sqrt{1-\rho^2}J_X \end{bmatrix}.$$

Substituting this result into (13), after some algebra, gives (14).

7.2 Sign of the square-root in (16)

With $\psi = 1$, as $\gamma \rightarrow 1$, the utility representation (8) converges to the log-utility representation. The optimal portfolio rule without ambiguity ($\eta = 0$), given by (22), in the special case of log-utility

($\gamma = \psi = 1$) is well-known (Merton (1969, 1971, 1973)):

$$\pi_t = (\mu - r) y_t,$$

i.e., the intertemporal hedging demand component disappears (if $\psi = \gamma = 1$, then $A = B = 0$). It is therefore necessary to guarantee that with $\psi = 1$, $\lim_{\gamma \rightarrow 1} A = 0$ ¹⁵ as the limit of (22) as $\gamma \rightarrow 1$ is given by:

$$\lim_{\gamma \rightarrow 1} \pi_t = (\mu - r) y_t + \left(\lim_{\gamma \rightarrow 1} A \right) \rho \sigma y_t.$$

From (16), $\lim_{\gamma \rightarrow 1} A$ is:

$$\lim_{\gamma \rightarrow 1} A = \frac{(\beta + \kappa) \pm \lim_{\gamma \rightarrow 1} (1 - \gamma) \gamma \sqrt{\left(\frac{\rho \sigma (\mu - r)}{\gamma} - \frac{\beta + \kappa}{1 - \gamma} \right)^2 - \frac{\sigma^2 (\mu - r)^2 [\gamma (1 - \rho^2) + \rho^2]}{\gamma^2 (1 - \gamma)}}{\sigma^2}. \quad (40)$$

If $\gamma \rightarrow 1^+$, i.e., $\gamma > 1$, then $(1 - \gamma) < 0$ and the discriminant of the square root in (40) is always > 0 . By assumption, $(\beta + \kappa) > 0$, which implies that, in order to have $\lim_{\gamma \rightarrow 1^+} A = 0$, the "+" sign must be considered.

The same rational implies that when $\gamma < 1$, the "-" sign of the square root guarantees $\lim_{\gamma \rightarrow 1^-} A = 0$ (it can be easily shown that the discriminant of the square root in (40) is positive as γ approaches 1 from below).

When $\gamma = 1$, only the "-" sign of the square root gives $A = 0$, as:

$$A |_{\gamma=1} = \frac{(\beta + \kappa) \pm (\beta + \kappa)}{\sigma^2}.$$

□

7.3 Optimal Consumption and Portfolio rules

7.3.1 Domain $\gamma \geq \omega$

The domain of analysis is set so that A in (16) is a real number, i.e., its discriminant is non-negative. Consequently the condition to be satisfied is

$$\left(\frac{\rho \sigma (\mu - r)}{\gamma} - \frac{\beta + \kappa}{1 - \gamma} \right)^2 \geq \frac{\sigma^2 (\mu - r)^2 [\gamma (1 - \rho^2) + \rho^2]}{\gamma^2 (1 - \gamma)}. \quad (41)$$

For $\gamma > 1$ it is straightforward to conclude that $\left(\frac{\rho \sigma (\mu - r)}{\gamma} - \frac{\beta + \kappa}{1 - \gamma} \right)^2 > \frac{\sigma^2 (\mu - r)^2 [\gamma (1 - \rho^2) + \rho^2]}{\gamma^2 (1 - \gamma)}$, and therefore (41) is always true.

¹⁵From (17) $\lim_{\gamma \rightarrow 1} A = 0 \implies \lim_{\gamma \rightarrow 1} B = 0$.

For $\gamma < 1$, (41) is true as long as:

$$\begin{aligned} \frac{\gamma}{1-\gamma} &\geq \frac{\sigma^2(\mu-r)^2}{(\beta+\kappa)^2} + \frac{2\rho\sigma(\mu-r)}{(\beta+\kappa)} \\ \Leftrightarrow \gamma &\geq \frac{\sigma^2(\mu-r)^2 + 2\rho\sigma(\mu-r)(\beta+\kappa)}{(\beta+\kappa)^2 + \sigma^2(\mu-r)^2 + 2\rho\sigma(\mu-r)(\beta+\kappa)} \\ \Leftrightarrow \gamma &\geq \omega, \end{aligned}$$

by making $\omega = \frac{\sigma^2(\mu-r)^2 + 2\rho\sigma(\mu-r)(\beta+\kappa)}{(\beta+\kappa)^2 + \sigma^2(\mu-r)^2 + 2\rho\sigma(\mu-r)(\beta+\kappa)}$. Note that $\omega < 1$ as $(\beta+\kappa)^2 > 0$.

The domain of analysis is therefore $\gamma \geq \omega$.

7.3.2 Optimal rules (19) and (20)

Considering the Bellman equation (14) and a value function with a structure as that of (15), the FOC with respect to π_t gives:

$$\pi_t = \frac{1}{\gamma + \sqrt{\frac{2\eta}{G(\pi)}}} \left[(\mu-r)y_t + \left(1 - \frac{1}{1-\gamma} \sqrt{\frac{2\eta}{G(\pi)}} \right) \sigma \rho y_t \frac{\partial g(y)}{\partial y} \right], \quad (42)$$

where:

$$G(\pi_t) = \frac{\pi_t^2}{y_t} + \frac{\sigma^2 y_t J_y^2}{X_t^2 J_X^2} + \frac{2\sigma \rho \pi_t J_y}{X_t J_X}, \quad (43)$$

i.e., optimal portfolio rule under ambiguity is the solution of an implicit function in π_t . In order to solve this optimization problem under ambiguity, following the perturbation theory under robust control problems, functions g and π are expanded in $(\sqrt{2\eta})$ to first order. Expansion of function g is given in (18) and expansion of function π_t is given by:

$$\pi(y_t, \sqrt{2\eta}) = \pi_0(y_t) + \pi_1(y_t) \sqrt{2\eta} + \mathcal{O}_\pi^2(\sqrt{2\eta}), \quad (44)$$

where $\mathcal{O}_\pi^2(\sqrt{2\eta})$ is a symbol representing terms of second order in $(\sqrt{2\eta})$. $\pi_0(y_t)$ represents the solution when there is no ambiguity ($\eta = 0$), with expression known from (22). The next step is to expand function G .

Making use of expanded functions $g(y_t, \sqrt{2\eta})$ and $\pi(y_t, \sqrt{2\eta})$, the expanded G function is obtained:

$$G(\pi_t(y_t, \sqrt{2\eta}), \sqrt{2\eta}) = G_0(y_t) + G_1(y_t) \sqrt{2\eta} + \mathcal{O}_G^2(\sqrt{2\eta}),$$

where following the same rational as before, $G_0(y_t)$ represents the solution without ambiguity. So the last step is to deduct the expression of $G_0(y_t)$. This is done by going back to (43) and consider

functions $g_0 (= Ay_t + B$ with A and B given by (16) and (17)) and π_0 , given by (22), which results in:

$$G_0(y_t) = \left[\left(\frac{(\mu - r) + \rho\sigma A}{\gamma} \right)^2 + \left(\frac{\sigma A}{1 - \gamma} \right)^2 + \frac{2\sigma\rho A((\mu - r) + \rho\sigma A)}{(1 - \gamma)\gamma} \right] y_t.$$

With this result and going back to (42), the asymptotic expansion for the optimal rule under ambiguity (20) is immediately obtained.

Regarding the optimal consumption rule (19), computations are more straightforward. Considering the Bellman (14) and the aggregator (8), the FOC with respect to variable C_t is simply:

$$f_C = J_X,$$

where f_C is the gradient of the aggregator (8) with respect to consumption. The optimal portfolio rule is given $C_t = \beta X_t$, not depending on the ambiguity parameter η . Its expansion in $(\sqrt{2\eta})$ is therefore simply given by:

$$C_t = \beta X_t + \mathcal{O}_c^2(\sqrt{2\eta}),$$

which is (19).

7.3.3 Optimal rules (27) and (28)

Considering the Bellman equation (26) and a value function with a structure as that of (15), the FOC with respect to π_t gives:

$$\pi_t = \frac{1}{\gamma}(\mu - r)y_t - \frac{\sqrt{1 - \rho^2}\sqrt{2\eta}}{\gamma}\sqrt{y_t} + \frac{\sigma\rho}{\gamma}\frac{\partial g(y)}{\partial y}y_t. \quad (45)$$

As for the general case in section 3, following the perturbation theory under robust control problems, the function g is expanded in $(\sqrt{2\eta})$ to first order. Expansion of function g is given in (18). Recalling from (18) that $g_0 (= Ay_t + B$ with A and B given by (16) and (17)) represents the function g when there is no ambiguity and going back to (45), the asymptotic expansion for the optimal rule under ambiguity (28) is immediately obtained.

Regarding the optimal consumption rule (27), considering the Bellman (26) and the aggregator (8), the FOC with respect to variable C_t is again given by:

$$f_C = J_X,$$

where f_C is the gradient of the aggregator (8) with respect to consumption. The asymptotic optimal portfolio rule is therefore given by:

$$C_t = \beta X_t + \mathcal{O}_c^2(\sqrt{2\eta}),$$

which is (27).

□

7.4 Bellman Equation (34)

Following the same approach as in appendix 7.1, Ξ represents the diffusion matrix of state variables y_t and X_t processes in (31) and (32):

$$\Xi = \begin{bmatrix} \sigma\sqrt{y_t}\rho & \sigma\sqrt{y_t}(1-\rho^2) \\ \pi_t\sqrt{\frac{1}{y_t}}X_t & 0 \end{bmatrix}. \quad (46)$$

The minimization of (33) with respect to the vector h gives:

$$\begin{bmatrix} h^s \\ h^\varepsilon \end{bmatrix} = -\frac{\sqrt{2\eta}}{\sqrt{\begin{bmatrix} J_y & J_X \end{bmatrix} \Xi \Xi^\top \begin{bmatrix} J_y \\ J_X \end{bmatrix}^\top}} \Xi^\top \begin{bmatrix} J_y \\ J_X \end{bmatrix}. \quad (47)$$

Replacing (46) in (47), the vector of optimal contaminating drifts is obtained and given by:

$$\begin{bmatrix} h^s \\ h^\varepsilon \end{bmatrix} = -\frac{\sqrt{2\eta}}{\sqrt{J_y^2\sigma^2y_t + 2J_yJ_X\sigma\rho\pi_tX_t + J_X^2\pi_t^2\frac{1}{y_t}X_t^2}} \begin{bmatrix} \sigma\sqrt{y_t}\rho J_y + \pi_t\sqrt{\frac{1}{y_t}}X_tJ_X \\ \sigma\sqrt{y_t}\sqrt{1-\rho^2}J_y \end{bmatrix}.$$

Substituting this result into (33), after some algebra, gives (34).

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