A Closed-Form Solution for Options with Ambiguity about Stochastic Volatility*

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Abstract

We derive a closed-form solution for the price of a European call option in the presence of ambiguity about the stochastic process that determines the variance of the underlying asset's return. The option pricing formula of Heston (1993) is a particular case of ours, corresponding to the case in which there is no ambiguity (uncertainty is exclusively risk). In the presence of ambiguity, the variance uncertainty price becomes either a convex or a concave function of the instantaneous variance, depending on whether the variance ambiguity price is negative or positive. We find that if the variance ambiguity price is positive, the option price is decreasing in the level of ambiguity (across all moneyness levels). The opposite happens if the variance ambiguity price is negative. Consistently, in the former (and more natural) scenario, ambiguity aversion decreases the option's implied volatility, which helps to explain the variance premium puzzle.

Keywords: Option Pricing, Stochastic Volatility, Ambiguity, Variance Premium Puzzle.

JEL Classification: C61 · D81 · G13.

1 Introduction

We derive a closed-form solution for the price of a European call option in the presence of ambiguity about the stochastic process that determines the variance of the underlying asset's return.

Since the seminal works of Black and Scholes (1973) and Merton (1973), the option pricing literature had an impressive development: models with stochastic variance of the underlying asset's return (e.g. Hull and White (1987), Johnson and Shanno (1987), Wiggins (1987), Scott (1987), Stein and Stein (1991), Heston (1993)), models with jumps in the underlying asset's price process with and without stochastic interest rate (e.g. Merton (1976), Bates (1996), Bakshi et al. (1997)), models with jumps in both the price and the variance processes (e.g. Duffie et al. (2000) and Barndorff-Nielsen and Shephard (2001)), jumps with finite activity (e.g. Bates (2000) and Pan (2002)) and infinite activity (e.g. Madan et al. (1998) and Carr and Wu (2003)), models using ARCH processes (e.g. Bollerslev and Mikkelsen

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(1996) and Heston and Nandi (2000)), among others. In Carr and Wu (2004) a general theoretical framework for "closed-form" option prices is provided, as well as a comprehensive survey of the option pricing literature.

The core of this literature focuses on the relationship between the underlying asset's price dynamics and the option price. This is understandable: that relationship is a central issue on option pricing, as an option represents a claim on the underlying asset. In this paper we explore a different angle of analysis: the nature of the uncertainty involved in the option pricing problem.

Due to the option's forward looking nature, uncertainty is intrinsically linked with its pricing. There are two standard approaches in the option pricing literature: no-arbitrage valuation and risk-neutral valuation. The former is based on obtaining a "riskless" portfolio excluding arbitrage possibilities between the option and the underlying asset, while the latter is centered in a change of probability measure, from the physical "real world" measure to an equivalent martingale "risk-neutral" measure under which the option price is obtained.[1] Independently of the approach, uncertainty is typically treated as risk, that is, as something that can be entirely described by a probability distribution.

But economic agents may not be able to completely describe the uncertainty that they face by using a single probability distribution. In this case, we say that agents face ambiguity. Essentially, risk refers to uncertainty that can be represented by a probability distribution, while ambiguity refers to uncertainty that cannot. This conceptual distinction was firstly pointed out by Knight (1921) and later supported by the empirical experiments of Ellsberg (1961) and others (see, for e.g., Camerer and Weber (1992) and Epstein and Schneider (2010) for a survey). This distinction has relevant implications for the behavior of economic agents, and, therefore, for economic theory in general. The rapidly growing literature on asset pricing under ambiguity has been comprehensively surveyed by Epstein and Schneider (2010).

Regarding the option literature, ambiguity is starting to be taken into account, even if there are still very few works and almost all of them are related to the exercise strategies of American options. Miao and Wang (2009) studied a real options problem, adopting a recursive multiple-priors utility model,[2] in which the source of ambiguity is the stochastic process which influences the continuation and termination payoffs associated with the option. They concluded that ambiguity lowers the option value. Kast et al. (2010) and Jaimungal (2010) have studied a similar problem regarding real options by using Choquet Brownian motions and a robust control approach, respectively.

Riedel (2009) developed a general theory of optimal stopping under time-consistent ambiguity in discrete time, using the recursive multiple-priors setting of Epstein and Schneider (2003). One of the applications is to American options, with the source of ambiguity being the payoff of exercising the option. This work was extended to a continuous-time environment in Riedel (2010). Chudjakow and Vorbrink (2009) used Riedel's (2009) framework to study the exercise strategies for several American exotic options under ambiguity. In Liu et al. (2005), ambiguity aversion about rare events in the economy's endowment was considered, generating an equilibrium premium for rare-events (which added to the standard risk premia for diffusive and jump risks, gives the equity premium in the model). The authors conclude that this ambiguity can help explain the well documented "smirk" pattern of option-

[1] When markets are incomplete, in the sense that a perfect hedge strategy is not available (as it is not possible to perfectly replicate the option using the available assets), there is an infinite number of equivalent martingale measures (EMM) under which the option price may be obtained (implying that the option price is not unique under incomplete markets). This is what happens when stochastic variance is considered (if there are not assets whose payoffs are contingent on the observed variance), where each of the EMM reflects a different variance uncertainty price. Superhedging, Mean-Variance hedging and Shortfall hedging are examples of hedging strategies under incomplete markets that have been studied in the literature (e.g. Cvitanic et al. (1999), Follmer and Sondermann (1986) and Follmer and Leukert (2000), respectively).

[2] An extensive review on decision theory under ambiguity has been carried out by Etner et al. (2012). Briefly, the two most common approaches being used in ambiguity literature are: the robust control (RC) approach, associated to an assumption of model uncertainty, and the multiple priors (MP) approach, whereby the single probability measure of the standard expected utility model is replaced by a set of probabilities or priors. The relationship between the robust control and multiple priors approaches has been widely discussed in the literature, for e.g., in Hansen and Sargent (2001), Hansen et al. (2002), Epstein and Schneider (2003), and Maccheroni et al. (2006).

implied volatility (e.g. Rubinstein (1994)).[3]

In this paper, we derive a closed-form solution for the price of a European call option when: (i) the underlying asset return's variance is stochastic and correlated with the spot asset return and (ii) there exists ambiguity about the variance stochastic process.

The major motivation for developing a stochastic variance option pricing model is the empirical evidence supporting the stochastic nature of risky assets return's variance (e.g. Eraker et al. (2003) and Eraker (2004)). Moreover, assuming stochastic variance allows to obtain more realistic return distributions, namely with higher kurtosis than that of the normal distribution (as assumed in Black and Scholes (1973)), non-zero skewness (negative skewness in case of a negative correlation between shocks in the return and in its variance, and positive skewness for positive correlation) and implied volatility surfaces closer to those observed in reality. It is also recognized in the literature (Bakshi et al. (1997)) that the most significant improvement over the model of Black and Scholes (1973) came from the introduction of stochastic variance. Once this is done, introduction of stochastic interest rates or jumps bring marginal improvements. The trade-off for obtaining a more realistic model is that stochastic variance option pricing models are more difficult to calibrate and, in most of the cases, are only approximately solved through time consuming numerical methods.

Regarding the source of ambiguity in our setting, the stochastic process of the variance, it has been advocated in the literature (Cao et al. (2005), Garlappi et al. (2007) and Ui (2011)) that it is reasonable to assume that investors estimate the variance of the risky asset's return without ambiguity, and that it is preferable to assume ambiguity about expected returns. Reasons invoked for this are analytical tractability, empirical evidence on the predictability of the variance of stock returns (Bollerslev et al. (1992)), higher difficulty in estimating the expected returns versus expected variance (Merton (1980)) and higher costs associated with errors in estimating expected returns versus expected variance (Chopra and Ziemba (1993)).

Nevertheless, we assume ambiguity about the stochastic process for the variance of the risky asset's return because: (1) the stochastic process of variance is a relevant option pricing input and there is no "a priori" reason to assume that investors are not ambiguous about it; (2) the expectation of variance under statistical-econometric methods isn't the sole relevant indicator of variance in the financial world, with the option-implied variance frequently differing both in level and dynamics from the statistical measure (e.g. Todorov (2010) and Drechsler and Yaron (2008)).

Our starting point is the stochastic variance option pricing model of Heston (1993), which is a well established model in the option pricing literature, offering a good trade-off between analytical and computational tractability and empirically realistic assumptions and results. Heston's (1993) option pricing model has a closed-form pricing formula,[4] without imposing any restriction regarding the correlation between the underlying asset return and its variance. Moreover, Heston's (1993) setting accounts for relevant stylized facts in financial data (apart from stochastic variance) as non-normal distribution of the assets returns, leverage effect (negative correlation between return and variance in some asset classes) and volatility clustering. Additionally, Black-Scholes option implied volatility surfaces generated by Heston's model are closer to those empirically observed (see, for e.g., Mikhailov and Nogel (2003)).

The option pricing formula obtained in this paper differs from that in Heston (1993) exclusively because a new specification for the variance uncertainty price is considered. Our variance uncertainty price specification takes into account the ambiguity about the variance stochastic process, being decomposed in two components: a variance risk price and a variance ambiguity price. It is shown that

[3]Liu et al. (2005) extended the economy of Lucas (1978) by considering rare events, which are modeled through a jump component in the economy's endowment process. Those rare events are the exclusive source of ambiguity. In their model, the price of the underlying asset of the European option follows a jump-diffusion process with no stochastic variance, and the European option pricing formula established in Merton (1976) is used.

[4]Rigorously speaking, the option pricing model of Heston (1993) delivers a semi closed-form solution, as it includes two integrals that cannot be evaluated exactly. They can however be approximated by using some numerical integration methods, e.g, Gauss-Lagendre or Gauss-Lobatto integration. Notwithstanding, Heston's option pricing formula is said to be a closed-form solution. The option pricing formula obtained in this paper will also be designated as a closed-form solution in this wider sense.

the model of Heston (1993) can be obtained as a particular case of our option pricing model, when ambiguity does not exist. The specification of the variance uncertainty price used in this paper is theoretically motivated by the general equilibrium model of Faria and Correia-da Silva (2012), where ambiguity is formally introduced through a "constraint preferences" robust control methodology. However, it is important to highlight that the option pricing formula obtained in the present paper is reached through an arbitrage approach and does not depend on the remaining assumptions in the model of Faria and Correia-da Silva (2012).

We therefore obtain a closed-form solution for the price of a European call option when the underlying asset return's variance is stochastic, correlated with the asset's spot return, and when there exists ambiguity about the stochastic process of the variance. This is the main result of the paper. We provide an illustration by simulating our option pricing model using the same calibration as Heston (1993).

We find that if the variance ambiguity price is positive, ambiguity about the variance's stochastic process leads to a decrease on option prices. Considering that a European option price (either call or put) is a positive function of variance, this means that our option pricing model helps to explain the variance premium puzzle. This puzzle refers to the fact that option-implied variance for a certain maturity tends to be higher than the conditional expectation of realized variance for that period of time (see, for e.g., Todorov (2010) and Drechsler and Yaron (2008)). When the variance ambiguity price is negative, the obtained option prices are increasing in the level of ambiguity.

Additionally, analyzing the impact of a correlation between shocks in the spot asset return and its variance, we conclude that when the variance ambiguity price is positive, ambiguity about the variance stochastic process implies, across all moneyness levels, a relative increase of option prices generated by our model versus those obtained under the Black and Scholes (1973) model with comparable variance. The opposite happens when the variance ambiguity price is negative.

In our view, this paper brings three major contributions: (1) it is the first time that ambiguity aversion within an option pricing problem with stochastic variance is considered, and where the latter's process is the source of ambiguity; (2) it is obtained a closed-form solution for an extension of the model of Heston (1993) which allows a non-linear specification for the variance uncertainty price; (3) in a specific scenario, it helps explaining the variance premium puzzle.

The paper is organized as follows. In section 2, the European call option closed-form solution is deducted. In section 3, simulation outputs are shown. In section 4, concluding remarks are made.

2 European Call Option Closed-Form Solution

It is assumed that the spot price, S_t, of the European call option's underlying asset evolves according to the following geometric Brownian motion:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_S, \tag{1}$$

where μ is the asset's expected return, v_t represents the instantaneous variance of the underlying asset's return and W_S is a standard Brownian motion. For simplicity, it is assumed, as in Heston (1993), that the underlying asset does not pay any income during the life of the option (e.g. stock with no dividends). However, it is well known that it is analytically straightforward to accommodate that feature (see, for e.g., Taylor (2005), chapter 14).

The instantaneous standard deviation, $\sqrt{v_t}$, is assumed to follow an Ornstein-Uhlenbeck process given by:

$$d\sqrt{v_t} = -\frac{\kappa}{2}\sqrt{v_t}dt + \frac{\sigma}{2}dW_v, \tag{2}$$

with $k > 0$, $\sigma > 0$ (with economic meaning given below) and W_v being a Brownian motion with an instantaneous correlation ρ with W_S, i.e., $dW_S dW_v = \rho dt$. From Itô's Lemma, the instantaneous variance, v_t, follows the process (Appendix 5.1):

$$dv_t = \left(\frac{\sigma^2}{4} - \kappa v_t\right) dt + \sigma\sqrt{v_t} dW_v, \tag{3}$$

which, letting $\theta = \frac{\sigma^2}{4\kappa}$ (as in Heston (1993)), can be rewritten as the following mean reverting square-root process:

$$dv_t = \kappa (\theta - v_t) dt + \sigma\sqrt{v_t} dW_v. \tag{4}$$

From (4) it results that k represents the variance mean-reversion parameter, θ is the expected value of variance and σ represents the standard deviation of variance. For simplicity, as in Heston's (1993) base case, a constant interest rate, r, is assumed.

In order to obtain the contingent claim pricing formula through standard arbitrage arguments (e.g. Black and Scholes (1973) and Merton (1973)), we need a specification for the variance uncertainty price. In the model of Heston (1993), where uncertainty is exclusively risk, the variance uncertainty price is proportional to the instantaneous variance. But the existence of ambiguity about the stochastic process of variance (4) motivates the use of a different variance uncertainty price specification, theoretically motivated by the general equilibrium model with ambiguity developed in Faria and Correia-da Silva (2012).[5] As ambiguity is the key issue of this paper, we describe briefly how ambiguity aversion about the stochastic process (4) is considered in Faria and Correia-da Silva (2012).

The stochastic process (4) evolves according to a probability measure, P, that describes the dynamics of v_t. In the presence of ambiguity about (4), an investor considers contaminations, P^h, around the reference belief, P. Those contaminations, representing alternative models for the dynamics of v_t, are assumed to be absolutely continuous with respect to P, and, therefore, are equivalently described by contaminating drift processes, h_v. Under each of the measures P^h, the Brownian motion is given by $W_v^h = W_v(t) + \int_0^t h_v(s)\, ds$.[6] Aversion towards ambiguity is introduced by assuming that, in the spirit of Gilboa and Schmeidler (1989), the representative investor bases his decisions on the worst possible contamination, i.e., the one associated with the lowest expected utility.[7]

As a result, under the investment opportunity set given by (1) and (4), from Faria and Correia-da Silva (2012), the variance uncertainty price specification to be used contains a new component that is not linear on v_t, being given by:

$$\lambda(v_t) = \underbrace{\lambda_1 v_t}_{risk\,price} + \underbrace{\lambda_2 \sqrt{v_t}}_{ambiguity\,price}, \tag{5}$$

[5]This is an intertemporal general equilibrium model based on the framework of Cox et al. (1985a) with two correlated state variables, a single production process and logarithmic utility. It is assumed that both state variables impact the expected output rate of the single production process in the economy, but only shocks in one of them are correlated with those in the output rate. Ambiguity about the stochastic process of that state variable is introduced, following the extension of the model of Cox et al. (1985b) made by Gagliardini et al. (2009). As an example, in Faria and Correia-da Silva (2012), we also deduct the equilibrium uncertainty price specification for a setting where unambiguous and ambiguous state variables follow the stochastic processes (1) and (4), respectively, and obtain the variance uncertainty price specification to be used in this paper.

[6]As explained by Gagliardini et al. (2009), the analysis is restricted, for tractability, to the class of Markov-Girsanov kernels $h_v(Y)$, where Y is the vector of state variables. Moreover, it is assumed an upper bound for the contaminating drift process h_v: $h_v^\top h_v \leqslant 2\eta$, where $\eta \geqslant 0$ is a parameter representing the level of ambiguity. This bound constrains both the instantaneous time variation and the continuation value of the relative entropy between the reference model, P, and any admissible contaminated model, P^h. This guarantees the rectangularity property of the set of priors (see Epstein and Schneider (2003) for the definition of this property and Trojani and Vanini (2004), p. 289, for a detailed explanation supporting the rectangularity property of the set of priors under the setting in Faria and Correia-da Silva (2012)).

[7]The approach of Gilboa and Schmeidler (1989), which is the most used in the literature on ambiguity, is sometimes criticized because it apparently implies extreme ambiguity aversion. However, the implied decision criteria may not be so extreme as it seems. The reasoning for this is that the set of priors is not an independent object including all logically possible priors, being instead part of the representation of the concrete problem under analysis. Klibanoff et al. (2005) developed a setting with smooth ambiguity aversion. However, there is still a debate in the literature about the axiomatic foundations of their model (see Epstein (2010) and Klibanoff et al. (2012) for a recent exchange on this).

where λ_1 and $\lambda_2 \in \mathbb{R}$.

Two comments on (5). First, without ambiguity ($\lambda_2 = 0$), the variance uncertainty price is proportional to the instantaneous variance, v_t, as in the model of Heston (1993). In the presence of ambiguity, the variance uncertainty price becomes either a concave or a convex function of the instantaneous level of variance, depending on whether the variance ambiguity price is positive ($\lambda_2 > 0$) or negative ($\lambda_2 < 0$). The second comment is that when the variance approaches zero ($v_t \to 0$), its uncertainty price also converges to zero, hence excluding arbitrage opportunities (see, for e.g. Cheridito et al. (2007), for a discussion on uncertainty price specifications and the existence of arbitrage opportunities).

Additionally, although the specification (5) is theoretically motivated by the model of Faria and Correia-da Silva (2012), it should be stressed that the contingent claim pricing results are obtained through a standard arbitrage approach, not depending on the remaining assumptions of that model.

We now have all the required inputs to deduce the closed-form solution for the price of an European call option when: (i) the dynamics of the underlying asset is given by (1) and (4); (ii) the representative investor is ambiguous about the stochastic process (4) of the underlying asset's return variance; and (iii) the investor is averse to ambiguity.

Considering the variance uncertainty price given by (5), applying Itô's lemma and making use of the standard arbitrage argument, it results that the price of a contingent claim $U(S_t, v_t, t)$ has to satisfy the following partial differential equation (PDE) (Appendix 5.2):

$$\begin{aligned}\frac{1}{2}v_t S_t^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma v_t S_t \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 U}{\partial v^2} + rS_t \frac{\partial U}{\partial S} & \\ + \left[\kappa(\theta - v_t) - \left(\lambda_1 v_t + \lambda_2 \sqrt{v_t}\right)\right]\frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} &= 0. \end{aligned} \tag{6}$$

We follow the methodology of Heston (1993), based on characteristic functions, to obtain the closed-form solution for a European call option with ambiguous stochastic variance.

A European call option with strike price K and maturing at time T satisfies the PDE (6) subject to the following boundary conditions:

$$\begin{aligned} U(S_T, v_T, T) &= Max(0, S_T - K), \\ U(0, v_t, t) &= 0, \\ \frac{\partial U}{\partial S}(\infty, v_t, t) &= 1, \\ rS_t \frac{\partial U}{\partial S}(S_t, 0, t) + \kappa\theta \frac{\partial U}{\partial v}(S_t, 0, t) - rU(S_t, 0, t) + \frac{\partial U}{\partial t}(S_t, 0, t) &= 0, \\ U(S_t, \infty, t) &= S_t. \end{aligned} \tag{7}$$

In the spirit of the Black and Scholes (1973) formula, our guess for the European call option price formula is

$$Call(S_t, v_t, t) = S_t P_1 - Ke^{-r(T-t)} P_2, \tag{8}$$

where $S_t P_1$ and $Ke^{-r(T-t)} P_2$ are the present value of the spot asset price upon option exercise and of the strike-price payment, respectively. Both of these terms have to satisfy (6). Substituting the guess formula (8) into (6), it results that both P_j ($j = 1, 2$) must satisfy the partial differential equations given by (Appendix 5.3):

$$\frac{1}{2}v_t \frac{\partial^2 P_j}{\partial x^2} + \sigma\rho v_t \frac{\partial^2 P_j}{\partial x \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 P_j}{\partial v^2} + (u_j v_t + r)\frac{\partial P_j}{\partial x} + \left(a - b_j v_t - \lambda_2\sqrt{v_t}\right)\frac{\partial P_j}{\partial v} + \frac{\partial P_j}{\partial t} = 0, \tag{9}$$

where $x_t = ln(S_t)$, $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, $a = \kappa\theta$, $b_1 = \kappa + \lambda_1 - \rho\sigma$ and $b_2 = \kappa + \lambda_1$.

The obtained PDEs (9) are subject to the terminal condition $P_j\left(x, v, T;\, ln(K)\right) = 1_{\{x_T \geqslant ln(K)\}}$ $(j = 1, 2)$, so that the option price satisfies the boundary conditions in (7).

A key issue in the option price formula (8) is to understand the nature of probabilities P_j $(j = 1, 2)$, and obtain their analytical expressions. For that, we start from the implied "uncertainty-neutral" processes of x_t and v_t:

$$\begin{aligned} dx_t &= \left(u_j v_t + r\right) dt + \sqrt{v_t} dW_S, & (10)\\ dv_t &= \left(a - b_j v_t - \lambda_2 \sqrt{v_t}\right) dt + \sigma \sqrt{v_t} dW_v, & (11)\end{aligned}$$

where the parameters u_j, r, a, b_j and σ are defined as before.

It is shown in Appendix 5.4 that, under the "uncertainty-neutral" processes (10)-(11), probabilities P_j $(j = 1, 2)$ are conditional probabilities of the option expiring in-the-money, under different probability measures:

$$P_j\left(x, v, T;\, ln(K)\right) = Pr\left[x_T \geqslant ln\left(K\right) \mid x_t = x,\, v_t = v\right],\; j = 1, 2. \tag{12}$$

From the guess formula (8), it is also clear that P_1 represents the delta of the European call option. However, as highlighted by Heston (1993), the probabilities P_j $(j = 1, 2)$ haven't a straightforward closed-form, but using the conditional characteristic functions of x for each $j = 1, 2$, denoted $f_j\left(x, v, T; \phi\right),\, j = 1, 2$, it is possible to obtain that closed-form. The characteristic functions $f_j\left(x, v, T; \phi\right),\, j = 1, 2$, continue to satisfy PDEs (9), subject to the terminal condition $f_j\left(x, v, T; \phi\right) = e^{i\phi x}$ (Appendix 5.4).

The characteristic function solution is:

$$f_j\left(x, v, t; \phi\right) = e^{C(\tau;\phi) + D(\tau;\phi) v_t + E(\tau;\phi)\sqrt{v_t} + i\phi x}\;,\; j = 1, 2\,, \tag{13}$$

where $\tau = T - t$ and

$$\begin{aligned} C\left(\tau; \phi\right) &= r\phi i\tau + \frac{a}{\sigma^2}\left[\left(b_j - \rho\sigma\phi i + d\right)\tau - 2ln\left(\frac{1 - ge^{d\tau}}{1 - g}\right)\right]\\ &\quad + \omega\left(\tau; \phi\right) - \Omega\,, & (14)\end{aligned}$$

$$D(\tau; \phi) = \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}\left(\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}\right), \tag{15}$$

$$E(\tau; \phi) = \frac{\left(b_j - \rho\sigma\phi i + d\right)\lambda_2}{\left(ge^{d\tau} - 1\right)\sigma^2 d}\left(4e^{\frac{d\tau}{2}} - 2e^{d\tau} - 2\right), \tag{16}$$

and

$$
\begin{aligned}
g &= \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}, \\
d &= \sqrt{\left(\rho\sigma\phi i - b_j\right)^2 - \sigma^2\left(2u_j\phi i - \phi^2\right)}, \\
\omega(\tau;\phi) &= \frac{-\left(b_j - \rho\sigma\phi i + d\right)\lambda_2^2}{2d^3\sigma^2}\left[\frac{\left(b_j - \rho\sigma\phi i + d\right)\left[1 - \left(4e^{\frac{d\tau}{2}} - 6\right)g - \left(4e^{\frac{d\tau}{2}} - 1\right)g^2\right]}{g^2\left(ge^{d\tau} - 1\right)}\right. \\
&\quad - \left(b_j - \rho\sigma\phi i - d\right)d\tau \\
&\quad - \frac{4\left[b(1-g) + d(1+g) + (g-1)\rho\sigma\phi i\right] ArcTanh\left(e^{\frac{d\tau}{2}}\sqrt{g}\right)}{g\sqrt{g}} \\
&\quad \left. + \frac{(1+g)\left[b(g-1) - d(g+1) - (g-1)\rho\sigma\phi i\right] ln\left(ge^{d\tau} - 1\right)}{g^2}\right], \\
\Omega &= \frac{-\left(b_j - \rho\sigma\phi i + d\right)\lambda_2^2}{2d^3\sigma^2}\left[\frac{\left(b_j - \rho\sigma\phi i + d\right)\left(1 + 2g - 3g^2\right)}{g^2(g-1)}\right. \\
&\quad - \frac{4\left[b(1-g) + d(1+g) + (g-1)\rho\sigma\phi i\right] ArcTanh\left(\sqrt{g}\right)}{g\sqrt{g}} \\
&\quad \left. + \frac{(1+g)\left[b(g-1) - d(g+1) - (g-1)\rho\sigma\phi i\right] ln(g-1)}{g^2}\right],
\end{aligned}
$$

where $ArcTanh$ represents the hyperbolic arc tangent of the complex number as argument.

It is straightforward to conclude that the model of Heston (1993) corresponds to the particular case of our setting in which $\lambda_2 = 0$, i.e., when there is no ambiguity and therefore uncertainty is exclusively risk (Appendix 5.5). This is consistent with the fact that, when $\lambda_2 = 0$, PDEs (6) and (9) are the ones in Heston (1993).

With the known characteristic functions $f_j(x, v, t; \phi)$, $j = 1, 2$, the expressions of probabilities P_j $(j = 1, 2)$, are obtained through the inverse Fourier transformation:

$$
P_j\left(x, v, T;\, ln(K)\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi ln[K]} f_j(x, v, T; \phi)}{i\phi}\right] d\phi\,,\ j = 1, 2. \tag{17}
$$

The closed-form solution for the European call option, under our setting, is therefore given by equations (8), (13) and (17).

3 Simulation Results

In this section, we analyze the effect on option prices of ambiguity about the stochastic process of the underlying asset return's variance.[8]

Under the uncertainty "neutralized" pricing probabilities, the underlying asset's return variance follows the square-root process:

$$
dv_t = \left[\kappa\theta - (\kappa + \lambda_1)v_t - \lambda_2\sqrt{v_t}\right]dt + \sigma\sqrt{v_t}dW_v\,. \tag{18}
$$

Model simulations are made under this process, and not the physical one (4). As known from the option pricing literature (e.g. Black and Scholes (1973) and Heston (1993)), no-arbitrage implies the

[8]Implementation using Dynare version 3.065 and MatLab version 7.0.0.19920 R14. The computation of the integrals in (17) is done through numerical integration, using an adaptive Gauss Lobatto rule.

irrelevance of the expected return of the underlying asset for option pricing, and therefore the process (18) exclusively determines option prices.

In the following simulation, the default calibration is the one used in Heston (1993). Two reasons motivate this choice. First, in this paper we are not focused on the estimation of the model and consequently we need to use some calibration from the literature.[9] Second, as our option price model is an extension of Heston (1993), his calibration is the natural candidate as it allows a comparative analysis. The default parameters values for the model implementation are presented in Table 1:

Table 1: Default Parameters Values

Parameter	**Value**
Mean Reversion - κ	2
Long-run variance - θ	0.01
Current variance - v_t	0.01
Std.Deviation of variance - σ	0.1
Option Maturity (years) - T	0.5
Interest rate - r	0
Strike Price - K	100
Correlation - ρ	0

The specification (5) for the variance uncertainty price implies that an additional parameter has to be calibrated: λ_2. We maintain $\lambda_1 = 0$, as in Heston (1993), which implies a null variance risk price. As a curiosity, under the equilibrium specification for λ_1 in the model of Faria and Correia-da Silva (2012), $\lambda_1 = 0$ when: (i) the variance is deterministic; (ii) the output rate of the economy is deterministic; or (iii) there is no correlation between shocks in the variance and in the output rate of the economy. In each of this scenarios, it is immediate to conclude that the motivation to hedge against adverse variance shocks (when they exist) disappears, which consistently implies a zero price for the variance risk.

With $\lambda_1 = 0$, we run simulations for different values of λ_2, which enables the isolation of the ambiguity effect on option pricing. The issue is how to calibrate λ_2. Alongside the well documented difficulties in estimating parameters in stochastic volatility models, the calibration of λ_2 has an extra difficulty: it economically embeds the ambiguity level of the representative investor, and estimating ambiguity is in itself a challenging task (see, for e.g., Maenhout (2006)).

In such context, we adopt a simple approach in order to get a qualitative perception of the impact of ambiguity about stochastic variance on option pricing: we simulate the option pricing model with arbitrary values for λ_2. In order to sort out those values, we consider the comment in Gagliardini et al. (2009) regarding the entropy-bound (η) for model contaminations, which is an indirect measure of the ambiguity level: it *"should not imply a too wide discrepancy between the reference model P and alternatives P^h"*. By other words, it should be a small number. Having said this, we simulate our option pricing model with three arbitrary values for λ_2: -0.02, 0, 0.02.[10] Heston's scenario is given by $\lambda_2 = 0$ (no ambiguity).

At a first stage, the simulation analysis is focused on the difference between the European call option prices obtained from our model and from the Black-Scholes (B-S) model with comparable volatility.[11] The volatility parameter used in the B-S model is obtained in the following way. The process (18) is

[9] There is an extensive literature on Heston's model calibration (e.g. Mikhailov and Nogel (2003) and Zhang and Shu (2003)).

[10] The η value implied by all calibrations in Gagliardini et al. (2009) is lower than 0.0136. Using the indicative specification $\lambda_2 = \pm\sigma\sqrt{2\eta\,(1-\rho^2)}$ from Faria and Correia-da Silva (2012), $\lambda_2 = \pm 0.02$ and $\sigma = 0.1$ (Table 1) imply $\eta = 0.02$ and $\eta = 0.027$, for $\rho = 0$ and $\rho = \pm 0.5$ respectively. With $\sigma = 0.2$, which we will use under some simulations, the implied η value is 0.005 and 0.007, for $\rho = 0$ and $\rho = \pm 0.5$ respectively.

[11] In the literature, the term "volatility" sometimes is used as "variance" and other times as "standard deviation". In this simulation, following Heston (1993) terminology, volatility is the square root of variance, i.e., means "standard deviation".

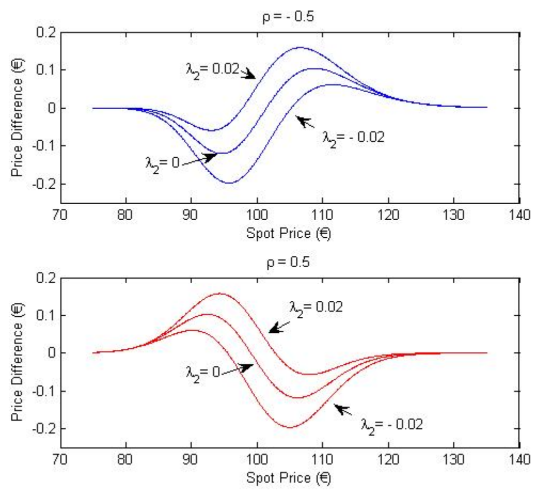
simulated n times (in our case $n = 100$), resulting from each simulation an average variance value. Then the average of those n average variance values is calculated. The square root of that average value is the volatility parameter input for the B-S option price computation.

The B-S model assumes constant variance and a normal distribution of the underlying asset return. However, as it is well documented in the literature, financial returns in general are skewed and show greater kurtosis that the normal distribution allows. The following analysis, as that of Heston (1993), is therefore centered in two parameters of the investment opportunity set that directly impact the skewness and the kurtosis of the underlying asset return distribution: ρ and σ, respectively.

We start by analyzing the difference between the option prices generated by our model and the B-S model, when a non-zero correlation ρ between shocks in variance and return is allowed. A positive value of ρ implies a positive skewness of the spot return distribution: if there is a higher variance when spot asset price rises, then a fatter right tail of the spot return distribution is generated. The opposite happens when $\rho < 0$.

For a complete spectrum of option moneyness (spot prices S from 75 to 135), Figure 1 discloses the option price differences when $\rho = -0.5$ and $\rho = 0.5$ and, for each of the ρ values, considering the three values for λ_2 $(-0.02, 0, 0.02)$.

Figure 1: Option Price Differences: ρ and λ_2 sensitivity.



Note: Price Difference is the difference between option prices from our stochastic variance model and from the Black-Scholes model with equal volatility to option maturity; Except for ρ and λ_2, parameter values are those in Table 1.

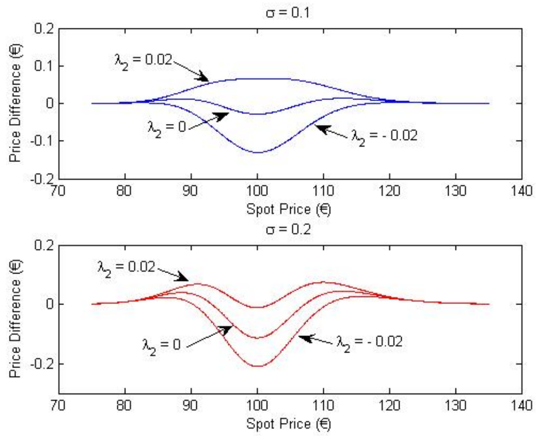
The first conclusion from Figure 1 is that, for both values of ρ, the introduction of ambiguity aversion about the stochastic variance process does not change the shape of the curve presented in Heston (1993) representing the differences between the option prices from a stochastic variance model and from the B-S model with comparable volatility. For a positive correlation ($\rho = 0.5$) between shocks in the spot asset return and in its variance, the prices of out-of-the-money (OTM) options are higher than those obtained in the B-S model, and lower for in-the-money (ITM) options. The contrary happens when the correlation is negative ($\rho = -0.5$).

However, Figure 1 also shows that, for both values of ρ and from far OTM to far ITM moneyness spectrum, when ambiguity is considered the graphical representation of option prices differences shifts upwards when $\lambda_2 = 0.02$ and downwards when $\lambda_2 = -0.02$. The conclusion is that ambiguity about the variance stochastic process implies a relative increase of option prices generated by our model, that accommodates for that ambiguity, versus B-S option prices with comparable volatility when the variance ambiguity price is positive (and the opposite when it is negative).

We now analyze the impact from changes in σ, the standard deviation of variance, on the difference between option prices obtained through our model and the B-S model with comparable volatility. When $\sigma = 0$, the variance is deterministic and spot returns have a normal distribution. When $\sigma > 0$, as it is assumed in our setting, the kurtosis of the spot return distribution increases.

We use the default parameters in Table 1 and one alternative value for σ $(= 0.2)$, for a complete spectrum of option moneyness (spot prices S from 75 to 135). For each of the σ values, the same three values for λ_2 $(-0.02, 0, 0.02)$ are considered. Simulation results are disclosed in Figure 2:

Figure 2: Option Price Differences: σ and λ_2 sensitivity.



Note: Price Difference is the difference between option prices from our stochastic variance model and from the Black-Scholes model with equal volatility to option maturity; Except for σ and λ_2, parameter values are those in Table 1.

It can be seen that when there is no ambiguity $(\lambda_2 = 0)$, as in Heston (1993), the price of far OTM and far ITM options increases versus B-S prices with comparable volatility, and the opposite happens for near-the-money options. Figure 2 also shows that when ambiguity is considered, for both σ values under analysis, there is a relative increase of option prices from far OTM to far ITM moneyness spectrum when the variance ambiguity price is positive $(\lambda_2 = 0.02)$ and the opposite when it is negative.

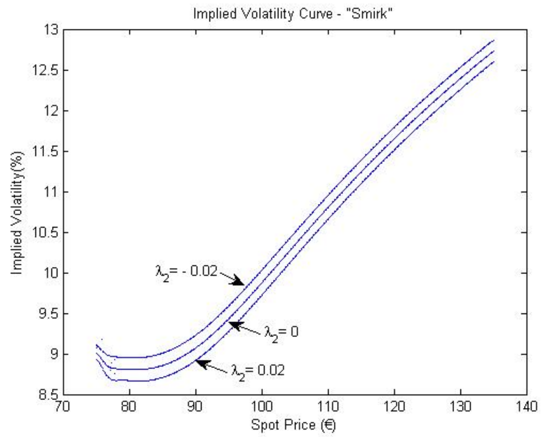
Note that in both Figure 1 and Figure 2 there is convergence between option prices from our stochastic variance model and the B-S model when the underlying asset spot price converges to zero or increases infinitely. This is expectable as, when the underlying asset price decreases, call price tends to zero, and when the underlying asset price increases, the call price tends to the difference between the underlying asset spot price and the option strike price (Rouah and Vainberg (2007), ch.10).

Until now, the analysis has been focused on the difference between option prices given by our model and by the B-S model with comparable volatility. However, as the same absolute price difference can have different meanings, it is convenient to analyze the implied volatility curve. This is done by making use of the one-to-one relationship between volatility and call option prices: option prices generated by our model are introduced in the B-S formula to obtain the implied volatility.

If $\rho < 0$, the underlying asset return distribution is negatively skewed which, in consistency with the option price differences disclosed in Figure 1, corresponds to an upwards "smirk" shape of the implied volatility curve from option prices. This happens when ambiguity is null. When ambiguity is considered, this upward "smirk" of the implied volatility curve remains unchanged but the curve moves slightly downwards (for all moneyness spectrum) when the variance ambiguity price is positive, and the contrary when it is negative, as it is illustrated in Figure 3.[12]

[12] When $\rho > 0$, the implied volatility curve has a downward "smirk" shape. Although not disclosed here, simulations run for $\rho = 0.5$ imply exactly the same conclusions as those obtained with $\rho = -0.5$ regarding the introduction of ambiguity.

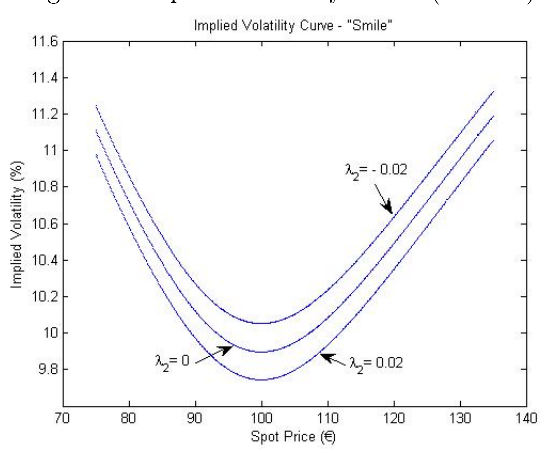
Figure 3: Implied Volatility Curve ($\rho = -0.5$)



Note: Implied volatility is obtained by considering the option price generated by our model into the B-S formula and then calculating the implied volatility input; Except for ρ and λ_2, parameter values are those in Table 1. The B-S volatility input with ($\lambda_2 = -0.02; \sigma = 0.1$) is 10.5%, with ($\lambda_2 = 0; \sigma = 0.1$) is 10% and with ($\lambda_2 = 0.02; \sigma = 0.1$) is 9.5%.

When $\sigma > 0$, kurtosis of the spot return distribution increases which, in consistency with the option price differences disclosed in Figure 2, corresponds to a "smile" shape of the implied volatility curve from option prices. This happens when ambiguity is null. When ambiguity is considered that "smile" shape is kept but the implied volatility curve moves slightly downwards (for all moneyness spectrum) when the variance ambiguity price is positive, and the contrary when it is negative. This is illustrated in Figure 4.

Figure 4: Implied Volatility Curve ($\sigma = 0.1$)



Note: Implied volatility is obtained by considering the option price generated by our model into the B-S formula and then calculating the implied volatility input; Except for λ_2, parameter values are those in Table 1. The B-S volatility input with ($\lambda_2 = -0.02; \sigma = 0.1$) is 10.5%, with ($\lambda_2 = 0; \sigma = 0.1$) is 10% and with ($\lambda_2 = 0.02; \sigma = 0.1$) is 9.5%.

From Figures 3 and 4 it is immediate to conclude that, with the variance uncertainty price specification (5), when ambiguity aversion about the variance stochastic process is considered option prices decreases versus a scenario with stochastic variance without ambiguity when the price for that ambiguity is positive and the contrary when it is negative. Moreover, considering the vertical scale of Figures 3 and 4 and the small shifts between implied volatility curves, we conclude that the magnitude of that price changes is small.

There is a "puzzle" in the literature concerning the fact that option-implied variance for a certain maturity tends to be higher than the conditional expectation of realized variance for that period of time (see, for e.g., Todorov (2010) and Drechsler and Yaron (2008)). This spread is commonly designated as the variance premium. This is an important puzzle to be solved, namely for the design of variance trading strategies (e.g. Bondarenko (2004) and Egloff et al. (2007)). The present paper gives a contribution in that direction. Concretely, under our option pricing formula when the variance ambiguity price is positive ($\lambda_2 > 0$), the option implied volatility decreases when ambiguity is taken into account. This means that the variance premium can effectively be smaller: the puzzle is partially explained by the consideration of ambiguity about the variance stochastic process.

Additionally, assume that the absolute value of the variance ambiguity price $\mid \lambda_2 \mid$ is a positive function of the level of ambiguity about the stochastic variance process.[13] When that ambiguity strongly increases, as it is expected to occur during periods of high "turbulence" in the financial markets, our option pricing formula implies (when $\lambda_2 > 0$) that the variance premium eventually disappears or even becomes negative. As shown in the next subsection, this seems to be consistent with the empirical evidence of the immediate months following the Lehman Brothers collapse in September 2008.

3.1 Variance Premium Data Analysis

Formally, the most common definition in the literature of the variance premium at date t (vp_t) is the spread of expectations under the uncertainty-neutral and the physical measures ($\mathbb{Q}$ and $\mathbb{P}$, respectively) of the variance of returns for the period $t+1$ (for simplicity we adopt a discrete time approach in this analysis):

$$vp_t = E_t^{\mathbb{Q}} v_{t+1} - E_t^{\mathbb{P}} v_{t+1}.$$

We follow the approach of Drechsler and Yaron (2008) for the estimation of vp and use data for both the VIX Index[14] and the S&P500 Spot Index from January 1^{st} 1990 (90m1) until November 26^{th} 2008 (08m11) from Bloomberg.

- $E_t^{\mathbb{Q}} v_{t+1}$: uses the VIX Index at date t, squaring its value (in order to be expressed in variance terms) and then dividing it by 12 (in order to get a monthly figure). The VIX value for a particular month is the value of the last observation for that month.

- $E_t^{\mathbb{P}} v_{t+1}$: uses the forecast at date t of the sum of the S&P500 Spot Index squared daily log-returns for the next month ($\hat{RV}_{t+1}$), obtained from the estimated regression of conditional variance given by $\hat{RV}_{t+1} = b_0 + b_1 * RV_t + b_2 * MA(1) + e_{t+1}$.[15] In Table 3 (Appendix 5.6) the estimation results for this regression are presented.

The (conditional) variance premium vp is therefore calculated on a monthly basis as:

$$vp_t = E_t^{\mathbb{Q}} v_{t+1} - E_t^{\mathbb{P}} v_{t+1} = \frac{VIX_t^2}{12} - \hat{RV}_{t+1},$$

with the key descriptive statistics being those in the following table:

[13] Which is an intuitive assumption and is theoretically supported by Faria and Correia-da Silva (2012).

[14] The VIX Index from CBOE is probably the most used volatility index, both in the literature and in the industry. It measures the one-month implied volatility in the S&P 500 Index option prices. For full details on the VIX Index construction methodology please see http://www.cboe.com/micro/vix/.

[15] In Drechsler and Yaron (2008), this is one of the three regressions of realized variance on lagged predictors. The other two regressions are based on squares of the log returns over 5-minute intervals during a month of the S&P500 Futures and S&P500 Index data (and also using a different regressor: the lagged VIX Index instead of $MA(1)$). The reason why we only use S&P500 Index daily data, is availability. The sample period in Drechsler and Yaron (2008) is monthly and covers 90m1 to 07m3.

Table 2: Descriptive Statistics for the Variance Premium (vp)

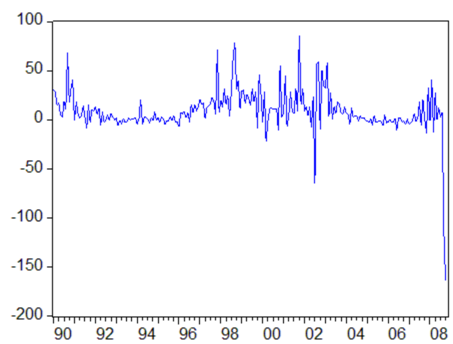
vp	$90m1 - 07m3$ - DY	$90m1 - 07m3$	$90m1 - 08m8$	$90m1 - 08m11$
Mean	12.67	12.71	12.21	8.93
Median	7.97	8.03	7.56	5.12
Minimum	-4.02	-4.11	-4.00	-162.69
Std. Deviation	14.38	14.40	14.08	22.47
Skewness	2.45	2.43	2.56	-2.03
Kurtosis	12.62	12.48	13.78	22.2

Note: Descriptive statistics in the column entitled $90m1 - 07m3$ - DY are those obtained in Drechsler and Yaron (2011) for the sample period $90m1 - 07m3$. Descriptive statistics in the remaining three columns are obtained for different sampling periods, as indicated in their respective titles.

The first sample period (90m1- 07m03) is used for comparison with the result of Drechsler and Yaron (2008), just to cross-check our vp estimations. Regarding the other two sampling periods, they differ by three months (September 08, October 08 and November 08), which are relevant months as they represent the immediate post-Lehman Brothers collapse time period. Uncertainty about the global financial system reached a very high level during that period, which Blanchard (2009) suggestively named as "Knight time". We therefore believe those are relevant months to test the hypothesis that when ambiguity strongly increases, the variance premium vp eventually disappears or even becomes negative.

Comparing the second and third columns in Table 2, it is clear that our vp computation is very close to that in Drechsler and Yaron (2008) (DY) for the same period $90m1 - 07m3$. Additionally, Table 2 allows to conclude that vp is relevant in absolute terms: average value of 9 percentage points between 90m1 to 08m11. More importantly for the analysis under way, vp statistics remain relatively unchanged when estimated using $90m1 - 07m3$ or $90m1 - 08m8$ data. However, they change significantly when the months from September to November 2008 are included.[16] In fact, as shown in Figure 5, between September and November 2008 the variance premium becomes strongly negative.

Figure 5: Variance Premium ($90m1 - 08m11$)



Note: Vertical axis is measured in percentage points. Horizontal axis respects to the time period: 1990m1 until 2008m11

This is also suggested by results in Table 2, by comparing the vp minimum value and the sign of its skewness under the last sampling period $90m1 - 08m11$ versus the others. Overall, this evidence supports the claim from our option pricing model that, if $\lambda_2 > 0$ and ambiguity increases, the variance

[16] This is formally confirmed with the rejection of the null hypothesis of no structural change in the estimated regression of conditional variance $\hat{RV}_{t+1} = b_0 + b_1 * RV_t + b_2 * MA(1) + e_{t+1}$, using the Chow breakpoint test at September 2008 (F-Statistic and Log likelihood ratio of 216.4193 and 310.5268, respectively).

premium can become negative.[17]

4 Concluding Remarks

We extended the option pricing model of Heston (1993) by considering ambiguity about the variance stochastic process. A new variance uncertainty price specification is used for the deduction of the option pricing formula. It contains two components: the variance risk price, which is proportional to the instantaneous variance, v_t, and the variance ambiguity price, which is proportional to the instantaneous standard-deviation, $\sqrt{v_t}$. This specification is theoretically motivated by the general equilibrium model in Faria and Correia-da Silva (2012). The model of Heston (1993) is obtained as a particular case when uncertainty is exclusively risk.

The main result of the paper is a closed-form solution for the price of a European call option when the underlying asset return's variance is stochastic, correlated with the asset spot return, and there exists ambiguity (aversion) about the variance stochastic process.

Analyzing the impact of correlation between shocks in the spot asset return and its variance and of the variance of variance, we conclude that when the variance ambiguity price is positive, ambiguity about the variance stochastic process implies, across all moneyness levels, a relative increase of option prices generated by our model versus those obtained under the Black and Scholes (1973) model with comparable variance. The opposite happens when the variance ambiguity price is negative.

Moreover, we conclude that when the variance ambiguity price is positive, ambiguity about the variance stochastic process induces a decrease on option prices versus a scenario with stochastic variance and no ambiguity. The contrary happens when the variance ambiguity price is negative.

The implied volatility curve from our model has a "smirk" shape, when the correlation between shocks in price and variance is non-zero, and a "smile" shape, when the variance of variance is non-zero. When the variance ambiguity price is positive, there is a downwards shift of the implied volatility curve when ambiguity is considered, for all option's moneyness spectrum. This means that, under that scenario for the variance ambiguity price, the developed option pricing model is a contribution for the explanation of the variance premium puzzle, particularly relevant for the design of variance trading strategies.

As future research topics on the back of this paper, we highlight the pricing and hedging of variance and volatility derivatives (including variance and volatility swaps) considering ambiguity about the stochastic variance process,[18] and the estimation of (i) the option's implied variance ambiguity price (adjusting the obtained pricing formula for the case in which the underlying asset pays some income during the life of the option) and (ii) the option's implied ambiguity level, studying its dynamics and empirical strength as market leading indicator (at least for the market of that option's underlying asset). We believe this latter information could be relevant not only for investors but also for policy makers.

[17] The sign of λ_2, and therefore the sign of the variance ambiguity price, needs to be estimated. This is left for future work.

[18] We thank Alejandro Balbás for this suggestion.

5 Appendices

5.1 Equation (3)

From (2) and recognizing that $v_t\left(\sqrt{v_t}, t\right) = \left(\sqrt{v_t}\right)^2$, application of Itô's Lemma (see, for e.g., Kamien and Schwartz (1991), chapter 22) gives:

$$\begin{aligned}
dv_t &= \left[\frac{\partial\left(\sqrt{v_t}\right)^2}{\partial t} + \frac{\partial\left(\sqrt{v_t}\right)^2}{\partial\sqrt{v_t}}\left(-\frac{\kappa}{2}\sqrt{v_t}\right) + \frac{1}{2}\frac{\partial^2\left(\sqrt{v_t}\right)^2}{\partial\sqrt{v_t}^2}\frac{\sigma^2}{4}\right]dt + \frac{\partial\left(\sqrt{v_t}\right)^2}{\partial\sqrt{v_t}}\frac{\sigma}{2}dW_v \\
\Leftrightarrow dv_t &= \left[0 + 2\sqrt{v_t}\left(-\frac{\kappa}{2}\sqrt{v_t}\right) + \frac{\sigma^2}{4}\right]dt + 2\sqrt{v_t}\frac{\sigma}{2}dW_v \\
\Leftrightarrow dv_t &= \left(-\kappa v_t + \frac{\sigma^2}{4}\right)dt + \sqrt{v_t}\sigma dW_v\,,
\end{aligned}$$

which is (3).

□

5.2 Equation (6)

The uncertainty neutral dynamics for S_t and v_t are given by:

$$dS_t = rS_t dt + \sqrt{v_t}S_t dW_S, \tag{19}$$

$$dv_t = \left[\kappa\left(\theta - v_t\right) - \left(\lambda_1 v_t + \lambda_2\sqrt{v_t}\right)\right]dt + \sigma\sqrt{v_t}dW_v, \tag{20}$$

where the drift in (19) is the risk free rate instead of the expected return μ as given in (1), in consistency with no-arbitrage arguments, and (20) has a drift adjustment given by the variance uncertainty price specification (5).

Applying Itô's Lemma, the dynamics of $U\left(S_t, v_t, t\right)$ are given by:

$$dU = \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial v}dv + \frac{\partial U}{\partial t}dt + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\left(dS\right)^2 + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\left(dv\right)^2 + \frac{\partial^2 U}{\partial S\partial v}dvdS, \tag{21}$$

where:

$$\begin{aligned}
\left(dS\right)^2 &= r^2S_t^2\left(dt\right)^2 + 2rS_t^2\sqrt{v_t}dtdW_S + v_tS_t^2\left(dW_S\right)^2 = 0 + 0 + v_tS_t^2dt\,; \\
\left(dv\right)^2 &= \left[\kappa\left(\theta - v_t\right) - \left(\lambda_1 v_t + \lambda_2\sqrt{v_t}\right)\right]^2\left(dt\right)^2 + \sigma^2 v_t\left(dW_v\right)^2 \\
&\quad + 2\left[\kappa\left(\theta - v_t\right) - \left(\lambda_1 v_t + \lambda_2\sqrt{v_t}\right)\right]\sigma\sqrt{v_t}dtdW_v = 0 + \sigma^2 v_t dt + 0\,; \\
dvdS &= rS_t\left[\kappa\left(\theta - v_t\right) - \left(\lambda_1 v_t + \lambda_2\sqrt{v_t}\right)\right]\left(dt\right)^2 + rS_t\sigma\sqrt{v_t}dtdW_v + \sigma v_tS_tdW_SdW_v \\
&\quad + \sqrt{v_t}S_t\left[\kappa\left(\theta - v_t\right) - \left(\lambda_1 v_t + \lambda_2\sqrt{v_t}\right)\right]dW_Sdt = 0 + 0 + \rho\sigma v_tS_tdt + 0.
\end{aligned}$$

Equation (21) can, therefore, be written as:

$$dU = \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial v}dv + \frac{\partial U}{\partial t}dt + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}v_tS_t^2dt + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\sigma^2 v_t dt + \frac{\partial^2 U}{\partial S\partial v}\rho\sigma v_tS_tdt. \tag{22}$$

The equity value of a hedged position is given by (see, for e.g., Black and Scholes (1973)):

$$S_t - \frac{U}{\frac{\partial U}{\partial S}}\,,$$

where $\frac{1}{\frac{\partial U}{\partial S}}$ is the number of options to buy. The change in the equity value of the hedged position (which is not a perfectly hedged position due to market incompleteness - see footnote 1) is given by:

$$dS - dU \frac{1}{\frac{\partial U}{\partial S}},$$

which from (22) equals:

$$-\left(\frac{\partial U}{\partial v} dv + \frac{\partial U}{\partial t} dt + \frac{1}{2}\frac{\partial^2 U}{\partial S^2} v_t S_t^2 dt + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\sigma^2 v_t dt + \frac{\partial^2 U}{\partial S \partial v}\rho\sigma v_t S_t dt\right)\frac{1}{\frac{\partial U}{\partial S}}.$$

Considering the no-arbitrage argument that the change in the equity value of the hedged position must be equal to the time effect $(r\, dt)$ on the value of that position, one obtains:

$$\begin{aligned}
\left(S_t - \frac{U}{\frac{\partial U}{\partial S}}\right) r dt &= -\left(\frac{\partial U}{\partial v} dv + \frac{\partial U}{\partial t} dt + \frac{1}{2}\frac{\partial^2 U}{\partial S^2} v_t S_t^2 dt + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\sigma^2 v_t dt + \frac{\partial^2 U}{\partial S \partial v}\rho\sigma v_t S_t dt\right)\frac{1}{\frac{\partial U}{\partial S}} \\
\Leftrightarrow \left(S_t \frac{\partial U}{\partial S} - U\right) r dt &= -\left(\frac{\partial U}{\partial v}\left[\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2 \sqrt{v_t})\right] dt + \frac{\partial U}{\partial v}\sigma\sqrt{v_t} dW_v + \frac{\partial U}{\partial t} dt \right. \\
&\quad \left. + \frac{1}{2}\frac{\partial^2 U}{\partial S^2} v_t S_t^2 dt + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\sigma^2 v_t dt + \frac{\partial^2 U}{\partial S \partial v}\rho\sigma v_t S_t dt\right).
\end{aligned}$$

Taking expectations on both sides of this equation ($E(dW_v) = 0$) and eliminating the common term dt, we obtain:

$$\frac{1}{2} v_t S_t^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma v_t S_t \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 U}{\partial v^2} + r S_t \frac{\partial U}{\partial S} + \left[\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2\sqrt{v_t})\right]\frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} = 0,$$

which is the partial differential equation (6).

□

5.3 Equation (9)

We prove that substituting (8) into (6) implies that probabilities P_j $(j = 1, 2)$ satisfy the PDEs (9). We analyze individually each of the terms in (8).

Starting with the first one and letting $U_1 = S_t P_1$, one gets:

$$\begin{aligned}
\frac{\partial U_1}{\partial S} &= P_1 + S_t \frac{\partial P_1}{\partial S}, \\
\frac{\partial^2 U_1}{\partial S^2} &= \frac{\partial P_1}{\partial S} + \frac{\partial P_1}{\partial S} + S_t \frac{\partial^2 P_1}{\partial S^2} = 2\frac{\partial P_1}{\partial S} + S_t \frac{\partial^2 P_1}{\partial S^2}, \\
\frac{\partial^2 U_1}{\partial S \partial v} &= \frac{\partial P_1}{\partial v} + S_t \frac{\partial^2 P_1}{\partial S \partial v}, \\
\frac{\partial U_1}{\partial v} &= S_t \frac{\partial P_1}{\partial v}, \\
\frac{\partial^2 U_1}{\partial v^2} &= S_t \frac{\partial^2 P_1}{\partial v^2}, \\
\frac{\partial U_1}{\partial t} &= S_t \frac{\partial P_1}{\partial t}.
\end{aligned}$$

As $U_1 = S_t P_1$ must satisfy the PDE (6), then:

$$\begin{aligned}
0 &= \frac{1}{2}v_t S_t^2 \left(2\frac{\partial P_1}{\partial S} + S_t \frac{\partial^2 P_1}{\partial S^2}\right) + \rho\sigma v_t S_t \left(\frac{\partial P_1}{\partial v} + S_t \frac{\partial^2 P_1}{\partial S \partial v}\right) + \frac{1}{2}\sigma^2 v_t S_t \frac{\partial^2 P_1}{\partial v^2} \\
&\quad + r S_t \left(P_1 + S_t \frac{\partial P_1}{\partial S}\right) + \left[\kappa\left(\theta - v_t\right) - \left(\lambda_1 v_t + \lambda_2 \sqrt{v_t}\right)\right] S_t \frac{\partial P_1}{\partial v} - r S_t P_1 + S_t \frac{\partial P_1}{\partial t} \\
\Leftrightarrow 0 &= v_t S_t \frac{\partial P_1}{\partial S} + \frac{1}{2} v_t S_t^2 \frac{\partial^2 P_1}{\partial S^2} + \rho\sigma v_t \frac{\partial P_1}{\partial v} + \rho\sigma v_t S_t \frac{\partial^2 P_1}{\partial S \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 P_1}{\partial v^2} + r S_t \frac{\partial P_1}{\partial S} \\
&\quad + \left[\kappa\left(\theta - v_t\right) - \left(\lambda_1 v_t + \lambda_2 \sqrt{v_t}\right)\right] \frac{\partial P_1}{\partial v} + \frac{\partial P_1}{\partial t} \\
\Leftrightarrow 0 &= \left(v_t + r\right) S_t \frac{\partial P_1}{\partial S} + \frac{1}{2} v_t S_t^2 \frac{\partial^2 P_1}{\partial S^2} + \rho\sigma v_t S_t \frac{\partial^2 P_1}{\partial S \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 P_1}{\partial v^2} \\
&\quad + \left[\rho\sigma v_t + \kappa\left(\theta - v_t\right) - \left(\lambda_1 v_t + \lambda_2 \sqrt{v_t}\right)\right] \frac{\partial P_1}{\partial v} + \frac{\partial P_1}{\partial t} \,. \qquad (23)
\end{aligned}$$

Making $x_t = ln\left(S_t\right)$, one obtains:

$$\begin{aligned}
\frac{\partial P_1}{\partial S} &= \frac{\partial P_1}{\partial x}\frac{dx}{dS} = \frac{\partial P_1}{\partial x}\frac{1}{S_t}, \\
\frac{\partial^2 P_1}{\partial S \partial v} &= \frac{1}{S_t}\frac{\partial^2 P_1}{\partial x \partial v}, \qquad (24) \\
\frac{\partial^2 P_1}{\partial S^2} &= -\frac{1}{S_t^2}\frac{\partial P_1}{\partial x} + \frac{1}{S_t}\frac{\partial^2 P_1}{\partial x^2}\frac{dx}{dS} = -\frac{1}{S_t^2}\frac{\partial P_1}{\partial x} + \frac{1}{S_t^2}\frac{\partial^2 P_1}{\partial x^2}\,.
\end{aligned}$$

With those results, equation (23) becomes:

$$\begin{aligned}
\left(v_t + r\right)\frac{\partial P_1}{\partial x} + \frac{1}{2}v_t \frac{\partial^2 P_1}{\partial x^2} - \frac{1}{2}v_t \frac{\partial P_1}{\partial x} + \rho\sigma v_t \frac{\partial^2 P_1}{\partial x \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 P_1}{\partial v^2} & \\
+ \left[\rho\sigma v_t + \kappa\left(\theta - v_t\right) - \left(\lambda_1 v_t + \lambda_2 \sqrt{v_t}\right)\right]\frac{\partial P_1}{\partial v} + \frac{\partial P_1}{\partial t} &= 0 \\
\Leftrightarrow \left(\frac{1}{2}v_t + r\right)\frac{\partial P_1}{\partial x} + \frac{1}{2}v_t \frac{\partial^2 P_1}{\partial x^2} + \rho\sigma v_t \frac{\partial^2 P_1}{\partial x \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 P_1}{\partial v^2} & \\
+ \left[\rho\sigma v_t + \kappa\left(\theta - v_t\right) - \left(\lambda_1 v_t + \lambda_2 \sqrt{v_t}\right)\right]\frac{\partial P_1}{\partial v} + \frac{\partial P_1}{\partial t} &= 0,
\end{aligned}$$

which is equation (9) when $j = 1$.

Regarding the second term in (8), the procedure is the same. Letting $U_2 = Ke^{-r(T-t)}P_2$, one gets:

$$\begin{aligned}
\frac{\partial U_2}{\partial S} &= Ke^{-r(T-t)}\frac{\partial P_2}{\partial S}, \\
\frac{\partial^2 U_2}{\partial S^2} &= Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial S^2}, \\
\frac{\partial^2 U_2}{\partial S \partial v} &= Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial S \partial v}, \\
\frac{\partial U_2}{\partial v} &= Ke^{-r(T-t)}\frac{\partial P_2}{\partial v}, \\
\frac{\partial^2 U_2}{\partial v^2} &= Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial v^2}, \\
\frac{\partial U_2}{\partial t} &= Kre^{-r(T-t)}P_2 + Ke^{-r(T-t)}\frac{\partial P_2}{\partial t}.
\end{aligned}$$

As $U_2 = Ke^{-r(T-t)}P_2$ must satisfy the PDE (6), then:

$$\begin{aligned} 0 &= \frac{1}{2}v_t S_t^2 K e^{-r(T-t)} \frac{\partial^2 P_2}{\partial S^2} + \rho\sigma v_t S_t K e^{-r(T-t)} \frac{\partial^2 P_2}{\partial S \partial v} + \frac{1}{2}\sigma^2 v_t K e^{-r(T-t)} \frac{\partial^2 P_2}{\partial v^2} \\ &\quad + r S_t K e^{-r(T-t)} \frac{\partial P_2}{\partial S} + \left[\kappa\left(\theta - v_t\right) - \left(\lambda_1 v_t + \lambda_2 \sqrt{v_t}\right)\right] K e^{-r(T-t)} \frac{\partial P_2}{\partial v} \\ &\quad - r K e^{-r(T-t)} P_2 + K r e^{-r(T-t)} P_2 + K e^{-r(T-t)} \frac{\partial P_2}{\partial t} \\ \Leftrightarrow 0 &= \frac{1}{2}v_t S_t^2 \frac{\partial^2 P_2}{\partial S^2} + \rho\sigma v_t S_t \frac{\partial^2 P_2}{\partial S \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 P_2}{\partial v^2} + r S_t \frac{\partial P_2}{\partial S} \\ &\quad + \left[\kappa\left(\theta - v_t\right) - \left(\lambda_1 v_t + \lambda_2 \sqrt{v_t}\right)\right] \frac{\partial P_2}{\partial v} + \frac{\partial P_2}{\partial t}. \end{aligned} \tag{25}$$

With $x_t = ln\left(S_t\right)$, and using results (24), equation (25) becomes:

$$\begin{aligned} 0 &= \frac{1}{2}v_t \left(\frac{\partial^2 P_2}{\partial x^2} - \frac{\partial P_2}{\partial x}\right) + \rho\sigma v_t \frac{\partial^2 P_2}{\partial x \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 P_2}{\partial v^2} + r\frac{\partial P_2}{\partial x} \\ &\quad + \left[\kappa\left(\theta - v_t\right) - \left(\lambda_1 v_t + \lambda_2 \sqrt{v_t}\right)\right] \frac{\partial P_2}{\partial v} + \frac{\partial P_2}{\partial t} \\ \Leftrightarrow 0 &= \frac{1}{2}v_t \frac{\partial^2 P_2}{\partial x^2} + \left(-\frac{1}{2}v_t + r\right)\frac{\partial P_2}{\partial x} + \rho\sigma v_t \frac{\partial^2 P_2}{\partial x \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 P_2}{\partial v^2} \\ &\quad + \left[\kappa\left(\theta - v_t\right) - \left(\lambda_1 v_t + \lambda_2 \sqrt{v_t}\right)\right] \frac{\partial P_2}{\partial v} + \frac{\partial P_2}{\partial t}, \end{aligned}$$

which is equation (9) when $j = 2$.

□

5.4 Characteristic Functions

We follow closely the explanation in Heston (1993), in order to: (i) prove that P_1 and P_2 are two conditional probabilities, under different measures, that the option expires in-the-money; (ii) obtain the expression of the characteristic function (13).

Assume that x_t and v_t follow the "uncertainty-neutral" processes (10)-(11). Consider a function $f\left(x, v, t\right)$ that is a conditional expectation of some other function g of the realizations of x and v at the maturity date T:

$$f\left(x, v, t\right) = E\left[g\left(x_T, v_T\right) \mid x_t = x,\ v_t = v\right]. \tag{26}$$

The terminal condition is implicit in the definition (26):

$$f\left(x, v, T\right) = g\left(x, v\right). \tag{27}$$

From Itô's Lemma, $f\left(x, v, t\right)$ dynamics is given by:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(dx\right)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\left(dv\right)^2 + \frac{\partial^2 f}{\partial x \partial v}dxdv. \tag{28}$$

Since $dW_S dW_v = \rho dt$, $dW_S dt = dW_v dt = 0$, $\left(dt\right)^2 = 0$, $\left(dW_S\right)^2 = \left(dW_v\right)^2 = dt$ and considering the "uncertainty-neutral" dynamics (10)-(11), one obtains:

$$\begin{aligned}
(dx)^2 &= (u_j v_t + r)^2 (dt)^2 + 2(u_j v_t + r)\sqrt{v_t} dt dW_S + v_t (dW_S)^2 = v_t dt\,, \\
(dv)^2 &= (a - b_j v_t - \lambda_2\sqrt{v_t})^2 (dt)^2 + 2(a - b_j v_t - \lambda_2\sqrt{v_t})\sigma\sqrt{v_t} dt dW_v + \sigma^2 v_t (dW_v)^2 = \sigma^2 v_t dt\,, \\
dx dv &= (u_j v_t + r)(a - b_j v_t - \lambda_2\sqrt{v_t})(dt)^2 + \sqrt{v_t}(a - b_j v_t - \lambda_2\sqrt{v_t}) dW_S dt \\
&\quad + (u_j v_t + r)\sigma\sqrt{v_t} dt dW_v + v_t \sigma dW_S dW_v = v_t \sigma\rho dt\,.
\end{aligned}$$

Introducing these results in equation (28), it becomes:

$$\begin{aligned}
df &= \left[\frac{\partial f}{\partial x}(u_j v_t + r) + \frac{\partial f}{\partial v}(a - b_j v_t - \lambda_2\sqrt{v_t}) + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} v_t \right. \\
&\quad \left. + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma^2 v_t + \frac{\partial^2 f}{\partial x \partial v} v_t \sigma\rho\right] dt + \frac{\partial f}{\partial x}\sqrt{v_t} dW_S + \frac{\partial f}{\partial v}\sigma\sqrt{v_t} dW_v. \qquad (29)
\end{aligned}$$

Considering that, by iterated expectations, $f(x, v, t)$ must be a martingale, then $E(df) = 0$. Taking expectations on both sides of equation (29), one obtains:

$$0 = \frac{1}{2} v_t \frac{\partial^2 f}{\partial x^2} + v_t \sigma\rho \frac{\partial^2 f}{\partial x \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 f}{\partial v^2} + (u_j v_t + r)\frac{\partial f}{\partial x} + (a - b_j v_t - \lambda_2\sqrt{v_t})\frac{\partial f}{\partial v} + \frac{\partial f}{\partial t}. \qquad (30)$$

We are now ready to obtain the desired proofs.

5.4.1 P_1 and P_2 are conditional probabilities that the option expires in-the-money

Comparing equations (9) and (30), it is immediate to conclude that, when x_t and v_t follow the uncertainty neutral processes (10)-(11), probabilities P_j $(j = 1, 2)$, and function f satisfy the same PDE.

Recalling the terminal condition imposed by (27), it is immediate to conclude that $g = 1$ if the option expires in the money and $g = 0$ otherwise, that is:

$$f(x, v, T) = g(x, v) = 1_{\{x \geqslant ln(K)\}}.$$

Then, the solution of (30) is the conditional probability at time t that the option expires in-the-money:

$$P_j(x, v, T;\, ln(K)) = Pr\left[x_T \geqslant ln(K) \mid x_t = x,\, v_t = v\right],$$

as we wanted to prove. Distinct dynamics under each of the uncertainty neutral measures corresponding to P_1 and P_2 result from differences between the specifications of u_j and b_j $(j = 1, 2)$ in (9), proved above (5.3), and also explained in Taylor (2005), p. 396.

□

5.4.2 Characteristic Function (13)

If $g(x, v) = e^{i\phi x}$, the solution of equation (30) is the characteristic function of $x = ln(S)$ (Heston (1993)). We guess the following functional form for the characteristic function:

$$f(x, v, t; \phi) = e^{C(T-t;\phi) + D(T-t;\phi)v_t + E(T-t;\phi)\sqrt{v_t} + i\phi x}. \qquad (31)$$

This guess is close to that of Heston (1993), with the difference being a new term $E(T - t; \phi)\sqrt{v_t}$ within the exponential argument. This new term results from the introduction of ambiguity aversion about the stochastic variance process.

The purpose is to obtain the expressions of $C(T-t;\phi), D(T-t;\phi)$ and $E(T-t;\phi)$. From (31):

$$\begin{aligned}
\frac{\partial f}{\partial x} &= i\phi e^{[\cdot]}, \\
\frac{\partial^2 f}{\partial x^2} &= -\phi^2 e^{[\cdot]}, \\
\frac{\partial f}{\partial v} &= \left(D + \frac{1}{2}\frac{1}{\sqrt{v_t}}E\right) e^{[\cdot]}, \\
\frac{\partial^2 f}{\partial v^2} &= \left[\left(D + \frac{1}{2}\frac{1}{\sqrt{v_t}}E\right)^2 - \frac{1}{4}\frac{1}{v_t\sqrt{v_t}}E\right] e^{[\cdot]}, \\
\frac{\partial^2 f}{\partial x \partial v} &= i\phi\left(D + \frac{1}{2}\frac{1}{\sqrt{v_t}}E\right) e^{[\cdot]}, \\
\frac{\partial f}{\partial t} &= \left(\frac{\partial C}{\partial t} + \frac{\partial D}{\partial t}v_t + \frac{\partial E}{\partial t}\sqrt{v_t}\right) e^{[\cdot]},
\end{aligned} \tag{32}$$

where $e^{[\cdot]} = e^{C(T-t)+D(T-t)v_t+E(T-t)\sqrt{v_t}+i\phi x}$.

Introducing those results in (30) and dividing by $e^{[\cdot]}$, one obtains:

$$\begin{aligned}
0 &= -\frac{1}{2}\phi^2 v_t + i\phi v_t \sigma\rho\left(D + \frac{1}{2}\frac{1}{\sqrt{v_t}}E\right) \\
&\quad + \frac{1}{2}\sigma^2 v_t\left[\left(D + \frac{1}{2}\frac{1}{\sqrt{v_t}}E\right)^2 - \frac{1}{4}\frac{1}{v_t\sqrt{v_t}}E\right] + i\phi\left(u_j v_t + r\right) \\
&\quad + \left(D + \frac{1}{2}\frac{1}{\sqrt{v_t}}E\right)\left(a - b_j v_t - \lambda_2\sqrt{v_t}\right) + \left(\frac{\partial C}{\partial t} + \frac{\partial D}{\partial t}v_t + \frac{\partial E}{\partial t}\sqrt{v_t}\right) \\
\Leftrightarrow 0 &= \left(i\phi r + Da + \frac{1}{8}\sigma^2 E^2 - \frac{1}{2}\lambda_2 E + \frac{\partial C}{\partial t}\right) \\
&\quad + \left(-\frac{1}{2}\phi^2 + i\phi D\sigma\rho + \frac{1}{2}\sigma^2 D^2 + i\phi u_j - Db_j + \frac{\partial D}{\partial t}\right) v_t \\
&\quad + \left(\frac{1}{2}Ei\phi\sigma\rho + \frac{1}{2}\sigma^2 DE - \frac{1}{2}Eb_j - \lambda_2 D + \frac{\partial E}{\partial t}\right)\sqrt{v_t} \\
&\quad + \left(\frac{1}{2}Ea - \frac{1}{8}\sigma^2 E\right)\frac{1}{\sqrt{v_t}}.
\end{aligned} \tag{33}$$

Equalizing the first coefficient and each of the others associated to v_t, $\sqrt{v_t}$, $\frac{1}{\sqrt{v_t}}$ to zero, equation (33) is equivalent to the system of the following four equations:

$$0 = i\phi r + aD + \frac{1}{8}\sigma^2 E^2 - \frac{1}{2}\lambda_2 E + \frac{\partial C}{\partial t} \tag{34}$$

$$0 = -\frac{1}{2}\phi^2 + i\phi u_j + i\phi\sigma\rho D - b_j D + \frac{1}{2}\sigma^2 D^2 + \frac{\partial D}{\partial t} \tag{35}$$

$$0 = \frac{1}{2}i\phi\sigma\rho E - \frac{1}{2}b_j E + \frac{1}{2}\sigma^2 DE - \lambda_2 D + \frac{\partial E}{\partial t} \tag{36}$$

$$0 = \frac{1}{2}\left(a - \frac{1}{4}\sigma^2\right)E. \tag{37}$$

Since $a = \kappa\theta$ (9) and $\theta = \frac{\sigma^2}{4\kappa}$ (4), equation (37) is always verified, and can therefore be ignored.

Consequently the relevant system to be solved in order to obtain the analytical expression of the characteristic functions is composed by the three differential equations (34)-(35)-(36) subject to $C(0;\phi) = D(0;\phi) = E(0;\phi) = 0$.

To solve this system, we start by noting that the non-linear differential equation (35) is the same as in Heston (1993), p. 341.[19] As this equation only depends on D, the solution of Heston (1993) for this equation continues to apply. We therefore have:

$$D(\tau;\phi) = \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}\left(\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right), \tag{38}$$

where $\tau = T - t$ and:

$$\begin{aligned} g &= \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}, \\ d &= \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2\left(2u_j\phi i - \phi^2\right)}. \end{aligned}$$

In order to obtain the expression for $E(\tau;\phi)$, we start by substituting the expression for $D(\tau;\phi)$ into equation (36):

$$\begin{aligned} 0 &= \frac{1}{2}i\phi\sigma\rho E - \frac{1}{2}b_j E + \frac{1}{2}\sigma^2\frac{(b_j - \rho\sigma\phi i + d)}{\sigma^2}\left(\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right)E \\ &\quad -\lambda_2\frac{(b_j - \rho\sigma\phi i + d)}{\sigma^2}\left(\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right) + \frac{\partial E}{\partial t} \\ \Leftrightarrow \frac{\partial E}{\partial \tau} &= \left[\frac{1}{2}(i\phi\sigma\rho - b_j) + \frac{1}{2}\frac{(b_j - \rho\sigma\phi i + d)\left(1-e^{d\tau}\right)}{(1-ge^{d\tau})}\right]E \\ &\quad -\lambda_2\frac{(b_j - \rho\sigma\phi i + d)\left(1-e^{d\tau}\right)}{\sigma^2\left(1-ge^{d\tau}\right)}, \end{aligned} \tag{39}$$

as $\tau = T - t \Rightarrow \frac{\partial E}{\partial \tau} = -\frac{\partial E}{\partial t}$.

Equation (39) is a standard linear non-homogeneous first order differential equation, whose solution is known to be of the type (Gothen (1997)):

$$E(\tau;\phi) = E_H(\tau;\phi) + E_P(\tau;\phi), \tag{40}$$

where $E_H(\tau;\phi)$ and $E_P(\tau;\phi)$ are the general solution of the homogeneous equation and a particular solution of the complete equation (39), respectively. Consider two new functions, $p(\tau;\phi)$ and $q(\tau;\phi)$, defined as:

$$\begin{aligned} p(\tau;\phi) &= -\left[\frac{1}{2}(i\phi\sigma\rho - b_j) + \frac{1}{2}\frac{(b_j - \rho\sigma\phi i + d)\left(1-e^{d\tau}\right)}{(1-ge^{d\tau})}\right], \\ q(\tau;\phi) &= \lambda_2\frac{(b_j - \rho\sigma\phi i + d)\left(1-e^{d\tau}\right)}{\sigma^2\left(1-ge^{d\tau}\right)}. \end{aligned}$$

Additionally, let $P(\tau;\phi)$ be a primitive of $p(\tau;\phi)$. Expressions for $E_H(\tau;\phi)$ and $E_P(\tau;\phi)$ are given by:

$$\begin{aligned} E_H(\tau;\phi) &= \vartheta e^{-P(\tau;\phi)}, \\ E_P(\tau;\phi) &= -e^{-P(\tau;\phi)}\left[\int_0^\tau e^{P(s;\phi)}q(s;\phi)\,ds\right], \end{aligned}$$

[19] Note that there is a typo in Heston (1993) regarding (35). See, for e.g., Mikhailov and Nogel (2003) for the correct expression (35).

with ϑ being an unknown constant.

To find the value of ϑ, use the restriction $E(0;\phi)=0$:

$$E\left(0;\phi\right) = -e^{-P(0;\phi)}\left[\int_0^0 e^{P(s;\phi)}q\left(s;\phi\right)ds\right]+\vartheta e^{-P(0;\phi)} = \vartheta e^{-P(0;\phi)} = 0 \Rightarrow \vartheta = 0.$$

Therefore:

$$E(\tau;\phi) = -e^{-P(\tau;\phi)}\left[\int_0^\tau e^{P(s;\phi)}q\left(s;\phi\right)ds\right]. \tag{41}$$

It remains to obtain the analytical expression of $P\left(\tau;\phi\right)$. One obtains:

$$\begin{aligned}
P\left(\tau;\phi\right) &= \int_0^\tau -\left[\frac{1}{2}\left(i\phi\sigma\rho-b_j\right)+\frac{1}{2}\frac{\left(b_j-\rho\sigma\phi i+d\right)\left(1-e^{ds}\right)}{\left(1-ge^{ds}\right)}\right]ds\\
\Leftrightarrow P\left(\tau;\phi\right) &= -\frac{1}{2}\left(i\phi\sigma\rho-b_j\right)\tau-\frac{1}{2}\int_0^\tau\frac{\left(b_j-\rho\sigma\phi i+d\right)\left(1-e^{ds}\right)}{\left(1-ge^{ds}\right)}ds\\
\Leftrightarrow P\left(\tau;\phi\right) &= -\frac{1}{2}\left(i\phi\sigma\rho-b_j\right)\tau-\frac{1}{2}\left[\left(b_j-\rho\sigma\phi i+d\right)\tau-2ln\left(\frac{1-ge^{d\tau}}{1-g}\right)\right]\\
\Leftrightarrow P\left(\tau;\phi\right) &= -\frac{1}{2}d\tau+ln\left(\frac{1-ge^{d\tau}}{1-g}\right).
\end{aligned}$$

This implies that:

$$e^{P(\tau;\phi)} = e^{-\frac{d\tau}{2}}\left(\frac{1-ge^{d\tau}}{1-g}\right) \Leftrightarrow e^{-P(\tau;\phi)} = e^{\frac{d\tau}{2}}\left(\frac{1-g}{1-ge^{d\tau}}\right).$$

Going back to (41) and substituting the obtained results we get:

$$\begin{aligned}
E(\tau;\phi) &= e^{\frac{d\tau}{2}}\left(\frac{g-1}{1-ge^{d\tau}}\right)\int_0^\tau e^{-\frac{ds}{2}}\left(\frac{1-ge^{ds}}{1-g}\right)\lambda_2\frac{\left(b_j-\rho\sigma\phi i+d\right)\left(1-e^{ds}\right)}{\sigma^2\left(1-ge^{ds}\right)}ds\\
\Leftrightarrow E(\tau;\phi) &= -e^{\frac{d\tau}{2}}\frac{\lambda_2\left(b_j-\rho\sigma\phi i+d\right)}{\sigma^2\left(1-ge^{d\tau}\right)}\int_0^\tau\left(e^{-\frac{ds}{2}}-e^{\frac{ds}{2}}\right)ds\\
\Leftrightarrow E(\tau;\phi) &= -e^{\frac{d\tau}{2}}\frac{\lambda_2\left(b_j-\rho\sigma\phi i+d\right)}{\sigma^2\left(1-ge^{d\tau}\right)}\left(-\frac{2e^{-\frac{d\tau}{2}}}{d}-\frac{2e^{\frac{d\tau}{2}}}{d}+\frac{4}{d}\right)\\
\Leftrightarrow E(\tau;\phi) &= \frac{\left(b_j-\rho\sigma\phi i+d\right)\lambda_2}{\left(ge^{d\tau}-1\right)\sigma^2 d}\left(4e^{\frac{d\tau}{2}}-2e^{d\tau}-2\right).
\end{aligned} \tag{42}$$

Expression (42) satisfies the condition $E(0;\phi)=0$.

Finally, going back to the equation (34), substituting the obtained expressions for $D\left(\tau;\phi\right)$ and

$E(\tau;\phi)$, and considering that $\frac{\partial C}{\partial \tau} = -\frac{\partial C}{\partial t}$, $C(\tau;\phi)$ is given by:

$$\begin{aligned}
C(\tau;\phi) &= \int_0^\tau \left(i\phi r + aD(\tau;\phi) + \frac{1}{8}\sigma^2 (E(\tau;\phi))^2 - \frac{1}{2}\lambda_2 E(\tau;\phi) \right) ds \\
\Leftrightarrow C(\tau;\phi) &= \int_0^\tau \left\{ i\phi r + a\frac{(b_j - \rho\sigma\phi i + d)}{\sigma^2} \left(\frac{1-e^{ds}}{1-ge^{ds}} \right) + \frac{1}{8}\sigma^2 \left[\frac{(b_j - \rho\sigma\phi i + d)\lambda_2}{(ge^{ds}-1)\sigma^2 d} \left(4e^{\frac{ds}{2}} - 2e^{ds} - 2 \right) \right]^2 \right. \\
&\quad \left. -\frac{1}{2}\lambda_2^2 \frac{(b_j - \rho\sigma\phi i + d)}{(ge^{ds}-1)\sigma^2 d} \left(4e^{\frac{ds}{2}} - 2e^{ds} - 2 \right) \right\} ds \\
\Leftrightarrow C(\tau;\phi) &= \int_0^\tau i\phi r + \frac{1}{4}(b_j - \rho\sigma\phi i + d) \left(\frac{1-e^{ds}}{1-ge^{ds}} \right) ds \\
&\quad + \int_0^\tau \frac{(b_j - \rho\sigma\phi i + d)\lambda_2^2 \left(2e^{\frac{ds}{2}} - e^{ds} - 1 \right)}{(ge^{ds}-1)\sigma^2 d} \left[\frac{(b_j - \rho\sigma\phi i + d)\left(2e^{\frac{ds}{2}} - e^{ds} - 1 \right)}{2(ge^{ds}-1)d} - 1 \right] ds.
\end{aligned}$$

We known (see above) that:

$$\int_0^\tau i\phi r + \frac{1}{4}(b_j - \rho\sigma\phi i + d) \left(\frac{1-e^{ds}}{1-ge^{ds}} \right) ds =$$

$$r\phi i\tau + \frac{1}{4}\left[(b_j - \rho\sigma\phi i + d)\tau - 2ln\left(\frac{1-ge^{d\tau}}{1-g} \right) \right].$$

So the remaining parcel to be simplified in order to obtain $C(\tau;\phi)$ is the following integral:

$$\begin{aligned}
&\int_0^\tau \frac{(b_j - \rho\sigma\phi i + d)\lambda_2^2 \left(2e^{\frac{ds}{2}} - e^{ds} - 1 \right)}{(ge^{ds}-1)\sigma^2 d} \left[\frac{(b_j - \rho\sigma\phi i + d)\left(2e^{\frac{ds}{2}} - e^{ds} - 1 \right)}{2(ge^{ds}-1)d} - 1 \right] ds \\
&= \frac{-(b_j - \rho\sigma\phi i + d)\lambda_2^2}{2d^3\sigma^2} \left[\frac{(b_j - \rho\sigma\phi i + d)\left[1 - \left(4e^{\frac{ds}{2}} - 6 \right) g - \left(4e^{\frac{ds}{2}} - 1 \right) g^2 \right]}{g^2 (ge^{ds}-1)} \right. \\
&\quad - (b_j - \rho\sigma\phi i - d)\, ds - \frac{4\left[b(1-g) + d(1+g) + (g-1)\rho\sigma\phi i \right] ArcTanh\left(e^{\frac{ds}{2}}\sqrt{g} \right)}{g\sqrt{g}} \\
&\quad \left. + \frac{(1+g)\left[b(g-1) - d(g+1) - (g-1)\rho\sigma\phi i \right] ln\left(ge^{ds} - 1 \right)}{g^2} \right]_0^\tau
\end{aligned}$$

where $ArcTanh\left(e^{\frac{ds}{2}}\sqrt{g} \right)$ denotes the hyperbolic arctangent of the complex number $e^{\frac{ds}{2}}\sqrt{g}$. Denoting:

$$\begin{aligned}
\omega(\tau;\phi) \equiv \frac{-(b_j - \rho\sigma\phi i + d)\lambda_2^2}{2d^3\sigma^2} &\left\{ \frac{(b_j - \rho\sigma\phi i + d)\left[1 - \left(4e^{\frac{d\tau}{2}} - 6 \right) g - \left(4e^{\frac{d\tau}{2}} - 1 \right) g^2 \right]}{g^2 (ge^{d\tau}-1)} \right. \\
&- (b_j - \rho\sigma\phi i - d)\, d\tau - \frac{4\left[b(1-g) + d(1+g) + (g-1)\rho\sigma\phi i \right] ArcTanh\left(e^{\frac{d\tau}{2}}\sqrt{g} \right)}{g\sqrt{g}} \\
&\left. + \frac{(1+g)\left[b(g-1) - d(g+1) - (g-1)\rho\sigma\phi i \right] ln\left(ge^{d\tau} - 1 \right)}{g^2} \right\}
\end{aligned}$$

and

$$\Omega \equiv \frac{-\left(b_j-\rho\sigma\phi i+d\right)\lambda_2^2}{2d^3\sigma^2}\left[\frac{\left(b_j-\rho\sigma\phi i+d\right)\left(1+2g-3g^2\right)}{g^2\left(g-1\right)}\right.$$
$$-\frac{4\left[b\left(1-g\right)+d\left(1+g\right)+\left(g-1\right)\rho\sigma\phi i\right]ArcTanh\left(\sqrt{g}\right)}{g\sqrt{g}}$$
$$\left.+\frac{\left(1+g\right)\left[b\left(g-1\right)-d\left(g+1\right)-\left(g-1\right)\rho\sigma\phi i\right]ln\left(g-1\right)}{g^2}\right] ,$$

one gets $[\omega(\tau;\phi)-\Omega]$ given by:

$$\int_0^\tau \frac{\left(b_j-\rho\sigma\phi i+d\right)\lambda_2^2\left(2e^{\frac{ds}{2}}-e^{ds}-1\right)}{\left(ge^{ds}-1\right)\sigma^2 d}\left[\frac{\left(b_j-\rho\sigma\phi i+d\right)\left(2e^{\frac{ds}{2}}-e^{ds}-1\right)}{2\left(ge^{ds}-1\right)d}-1\right]ds.$$

We can now write the analytical expression for $C(\tau;\phi)$:

$$\begin{aligned}
C(\tau;\phi) &= \int_0^\tau i\phi r+\frac{1}{4}\left(b_j-\rho\sigma\phi i+d\right)\left(\frac{1-e^{ds}}{1-ge^{ds}}\right)ds \\
&+\int_0^\tau \frac{\left(b_j-\rho\sigma\phi i+d\right)\lambda_2^2\left(2e^{\frac{ds}{2}}-e^{ds}-1\right)}{\left(ge^{ds}-1\right)\sigma^2 d}\left[\frac{\left(b_j-\rho\sigma\phi i+d\right)\left(2e^{\frac{ds}{2}}-e^{ds}-1\right)}{2\left(ge^{ds}-1\right)d}-1\right]ds\,, \\
\Leftrightarrow C(\tau;\phi) &= r\phi i\tau+\frac{1}{4}\left[\left(b_j-\rho\sigma\phi i+d\right)\tau-2ln\left(\frac{1-ge^{d\tau}}{1-g}\right)\right]+\omega(\tau;\phi)-\Omega.
\end{aligned}$$

Note that the condition $C(0;\phi)=0$ is satisfied, as $\omega(0;\phi)=\Omega$.

□

5.5 The model of Heston (1993): a Particular Case

If $\lambda_2 = 0$, i.e., there is no ambiguity, then from (42), $E(\tau;\phi)=0$ for any τ. With $E(\tau;\phi)=0$, the differential equation (34) becomes:

$$i\phi r+aD+\frac{\partial C}{\partial t} = 0\,,$$

the same obtained by Heston (1993) and from which his expression of $C(\tau;\phi)$ is derived.

Our expression for $D(\tau;\phi)$ is the same as in Heston (1993), as previously stated - see comment before equation (38).

So when $\lambda_2=0$, $C(\tau;\phi)$and $D(\tau;\phi)$ are the same as in Heston (1993), and $E(\tau;\phi)$ becomes zero.

It remains to examine the expression for $f_j(x,v,t;\phi)$ when $\lambda_2=0$. It is immediate to conclude from (13) that, in this scenario, it becomes $f_j(x,v,t;\phi)=e^{[C(\tau;\phi)+D(\tau;\phi)v_t+i\phi x]}$, which is the same as in Heston (1993).

It is therefore proved that Heston (1993) model can be obtained as a particular case of our setting when there is no ambiguity ($\lambda_2=0$).

□

5.6 Conditional Expectations of Realized Variance: Estimations

Table 3: Conditional Realized Variance

Sample Period	b_0	b_1	b_2	R^2
$90m1 - 07m3$	20.59	0.82	-0.34	0.40
$(t - stat)$	(4.81)	(13.26)	(-3.43)	
$90m1 - 08m8$	21.81	0.83	-0.39	0.39
$(t - stat)$	(5.20)	(14.53)	(-4.11)	
$90m1 - 08m11$	27.37	0.49	0.50	0.55
$(t - stat)$	(4.10)	(5.50)	(5.30)	

Note: This table presents our estimations results of the regression for the conditional realized variance: $\hat{RV}_{t+1} = b_0 + b_1 * RV_t + b_2 * MA(1) + e_{t+1}$. Results are disclosed for the three sampling periods considered. OLS method is used.

□

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