Asset Pricing with Fixed Asset Supply

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Abstract

This paper argues that equilibrium restrictions play a far more important role in understanding asset price behavior than is commonly considered. To illustrate this, we develop a general equilibrium dynamic asset pricing model in continuous time with two main features. First, asset supply (the number of outstanding shares) is fixed. Second, we impose market clearing conditions and characterize the price process that is viable under these restrictions. While our model contains only a few parameters that are fixed over time, we show that the resulting price process matches many stylized facts from the empirical literature (e.g. fat tails, GARCH, varying risk premium, mean reversion, time-varying betas, value and size effects). In addition, the return process we derive is intuitive and parsimoneous, and lends itself well for empirical applications.

Key words: asset pricing, general equilibrium, two-fund separation, return predictability
JEL: C22, D53, G11, G12, G14

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1 Introduction

As the recent literature on asset pricing has pointed out (cf. Cochrane 2011, Campbell and Cochrane 2000; Cochrane, 2001; Campbell, 2003), a successful consumption based asset pricing model should explain a high level and volatility of stock returns, with low and relatively constant interest rates, consumption growth with small volatility, and time-varying risk premiums; in particular it should explain why low prices predict high expected excess returns. In addition, it should explain the time-varying volatility of stock returns and calibration of the model should not lead to absurdly high risk aversion. Moreover, it should be able to explain why there exists differences in the cross-section of returns related to value or growth, and it should exhibit mean-reverting betas. In this paper we argue that general equilibrium restrictions and an (approximately) constant number of shares are crucial ingredients for an asset pricing model to generate these properties.[1] To illustrate our point we formulate a simple asset-pricing model that is able to explain all of these stylized facts.

As has been pointed out in recent years by for example Cochrane (2001,2008) and Raimondo (2005), and earlier by for example Bossaerts (1988) and Fernholz and Shay (1982), and to our knowledge for the first time by Rosenberg and Ohlsen (1976), i.i.d. returns are not compatible with a fixed supply of assets. If returns are i.i.d., the two-fund separation principle tells us that the main reason for investors to change their share of wealth in various assets disappears. This means that over time the investor's desired shareholdings stay in fixed proportions. So if prices are to change, this must be a result of changes in asset supply. When asset supply is fixed, however, it follows that returns cannot be i.i.d. unless they are perfectly correlated. Initially, Merton (1975) discarded the argument by Rosenberg and Ohlsen, mentioning that asset supply adjust through stock repurchases/issues and dividend payments. The problem was subsequently addressed occasionally, but generally ignored after Merton's rebuttal. However, the stylized

[1] We focus exclusively on fully rational representative agents. Recent developments in evolutionary finance (cf. Hommes and Wagener, 2009) are also able to explain many of the stylized facts.

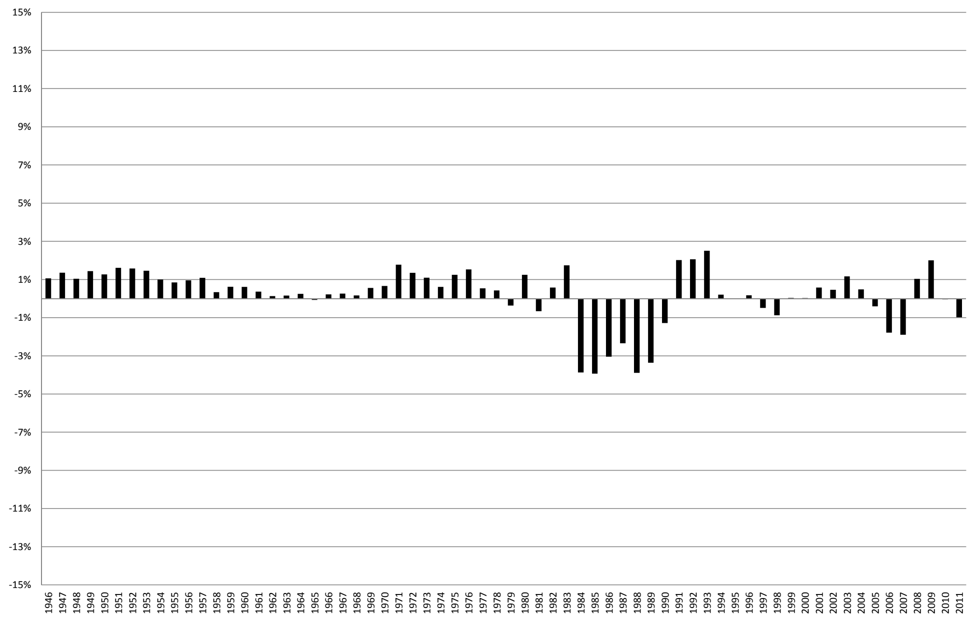

Figure 1. Net in and outflows of Corporate Equity in the U.S. for 1946-2011
Source: Flow of Funds Accounts, Federal Reserve.

facts reveal that a zero transaction assumption is more realistic than assuming that there are substantial transactions. Figure 1 shows the net percentage yearly in and outflows of corporate equity in the U.S. for 1946-2011. The yearly change in corporate equity is 0.22% on average with a standard deviation of 1.46%. Dichev (2007) also shows that the historical net in- and outflows of assets (both in terms of dividends and stock repurchases) hardly ever exceed 5% of total market capitalization.

The fundamental issue is straightforward: we cannot all rebalance. The average investor must hold the market portfolio. Unless asset supply is perfectly elastic, we cannot simply assume an exogenously given price process, since prices must be such that markets clear. Our model is one of the few attempts to address this issue; instead of taking the return process as given, we endogenously solve for an equilibrium price process by clearing markets and characterize the resulting return process. To provide a clear intuition, we keep the model simple so that we find closed-form solutions. We then discuss the properties of the return process. It turns out that the qualitative behavior of the equilibrium return process is remarkably similar to actual stock returns and we show that it matches many stylized facts.

Hence, our contribution is not so much to theoretically (re)address the equilibrium restrictions, but mainly to stress the empirical importance of the implications of the equilibrium restrictions.

Cochrane et al.(2008) also address market clearing restrictions. They develop a two-tree Lucas (1978) asset pricing model and show that the price and return dynamics correspond to many stylized facts. Recently, this approach has been generalized to multiple stocks in a "Lucas Orchard" (Martin, 2012). The difference of our approach with Cochrane et al. (2008) and Martin (2012), is that we do not introduce stochastic shocks through (high frequency) dividend payments. The advantage of not having dividends is that we derive expressions that are much more intuitive and parsimonious, and lend themselves well for empirical applications. At the same time, we think the drawback of not having dividend payments in the model is limited, since i) we empirically observe that dividends are not payed out on a monthly basis, let alone on a daily basis ii) dividend payments are only a fraction of the total returns while capital gains remain to comprise the lion's part of stock returns, iii) in a model with a frictionless market, Modigliani-Miller will hold. From the perspective of economic theory, the two approaches are equivalent.

We are also not the first to study the restrictions on the set of price processes that constitute a market equilibrium when there is a fixed number of assets and no dividends. [2] For an economy with a single risky asset, such restrictions have been analyzed by e.g. Bick (1990), He and Leland (1993), and Raimondo (2005). Bick (1990) studies viable price processes in an economy in which the representative agent only consumes in the terminal period. He derives a necessary and sufficient equilibrium condition in the form of a partial differential equation for time-homogeneous price processes. He and Leland (1993) generalize the results of Bick (1990) and derive a similar

[2] Though any arbitrage free price system can in principle constitute a market equilibrium, the focus of the literature has been on market equilibria resulting from "well-behaved" preferences, e.g. von Neumann-Morgenstern preferences. Of course if one allows preferences to be state and/or time dependent, all one really requires is a no arbitrage restriction; preferences can then always be defined such that they are consistent with the price system.

condition for general diffusion processes. Raimondo (2005) takes a somewhat different approach, he also discusses an economy in which consumption takes place only in the terminal period, but models the risky stock as a claim on an asset which pays an uncertain amount only in the terminal period. This approach is essentially equivalent to the former two studies. He also observes that the resulting price process can in general not be geometric Brownian motion.

Our contribution to the literature is three-fold. First, we allow for intermediate consumption in the model and show that this does not alter the set of admissible price processes. Second, instead of focusing on a single risky asset, we derive price processes for an economy with multiple risky assets. The extension to multiple assets is not innocent; while for certain preferences (power and log utility) one can show that geometric Brownian motion is compatible with market equilibrium when there is one asset, for the multiple asset case geometric Brownian motion is no longer compatible with equilibrium for these preferences. Third, and foremost, the articles mentioned above are primarily concerned with the existence and characterization of equilibrium price paths and spend little attention to the implications and properties of the resulting price process; we focus on an explicit derivation in terms of stock returns and confront the equilibrium return process with the stylized facts.

The outline of the paper is as follows. In Section 2, the model is presented and the return process is derived. In Section 3, we discuss the properties of the price process in light of existing empirical evidence. Section 5 concludes.

2 The model

We present a standard stock market model with a fixed interest rate and a risk-free asset (a bond or a bank account) and N risky assets. We assume a representative investor with von Neumann-Morgenstern preferences. The representative investor's expected lifetime utility over a finite period T is given by:

$$E_0 \int_0^T u(c(t))e^{-\rho t}dt, \quad (1)$$

where $u(c_t)$ is the felicity function, c_t consumption at time t, and ρ the discount factor. The investor maximizes (1) subject to:

$$c(t)dt = -B(t)dn(t) - \sum_{i=1}^{N} p_i(t)dm_i(t) \quad \text{for } 0 < t < T, \quad (2)$$

$$c(T) = B(T)n(T) + \sum_{i=1}^{N} p_i(T)m_i(T), \quad (3)$$

where $n(t)$ and $m_i(t)$ are the number of bonds and shares of stock i held by the investor at time t, $B(t)$ and $p_i(t)$ are the price of the bond and stock at time t. The investors initial wealth is equal to $n(0)B(0) + \sum_{i=1}^{N} m_i(0)p_i(0)$ which is exogenously given. We assume a fixed interest rate r, so that the dynamics of the bond price are given by:

$$dB(t) = rB(t) \quad (4)$$

We assume that the price process of each stock $p_i(t)$ is a diffusion process and characterized by:

$$dp_i(t) = \mu_i(p_1(t), \ldots, p_N(t), B(t), t)dt + \sigma_i(p_1(t), \ldots, p_N(t), B(t), t)dz_i(t) \quad (5)$$

where $\mu_i(\cdot)$, and $\sigma_i(\cdot)$ are to be determined by market equilibrium, but we assume that they are twice continuously differentiable with respect to $p_i(t)$ and $B(t)$, and continuously differentiable with respect to t. Note that our approach differs here from most asset pricing models; commonly the functions of the drift and diffusion are exogenously specified, while we treat them as endogenous. The process $z_i(t)$ is standard Brownian motion, and we allow for correlation between the Brownian motions of each asset, so that:

$$dz_i(t)dz_j(t) \equiv \rho_{ij}dt, i \neq j \quad (6)$$

Solving the maximization problem, we find the first-order condition for the

i-th stock price at time t:[3]

$$p_i(t) = e^{-\rho(T-t)} \frac{E_t[u'(c(T))p_i(T)]}{u'(c(t))}. \tag{7}$$

For the bond we find:

$$B(t) = b(T)e^{-\rho(T-t)} \frac{E_t[u'(c(T))]}{u'(c_t)}. \tag{8}$$

This gives us the price of asset i at time t:

$$p_i(t) = \frac{B(t)}{B(T)} \frac{E_t[u'(c(T))p_i(T)]}{E_t[u'(c(T))]} \tag{9}$$

Note that, given that the expectations exist and the resulting price process is well-behaved, equation (9) holds for general utility and general stochastic processes that are a martingale.

The set up so far and the solution to the maximization problem is quite standard, however, we impose the restriction that the number of shares $m_i(t)$ and bonds $n(t)$ are fixed and solve for the equilibrium price process such that the representative investor is in zero net supply each period. So instead of assuming prices are predetermined, quantities of shares are predetermined and prices must adjust so that the stock market is in equilibrium. This yields restrictions on the price paths in terms of $\mu_i(\cdot)$, and $\sigma_i(\cdot)$. Once we have an expression for the stochastic price paths, we can write down the distribution of returns. [4]

The conditional expectations in (9) are generally hard to evaluate, let alone checking whether the restriction on the price process holds, but with Constant Absolute Risk Aversion (CARA) utility and a diffusion process in the form of equation (5) we can obtain a closed form solution for the price process that satisfies equilibrium. [5]

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[3] For a derivation, see the appendix

[4] This is similar to Lucas (1978), however, in our specification capital is not leaking out of the market via dividend payments.

[5] For an interest rate equal to zero we also provide a closed-form solution for Hyperbolic Absolute Risk Aversion (HARA) in the appendix. HARA utility is fairly general, it nests risk-neutral preferences (linear utility), mean-variance preferences

Proposition 1 *Consider a representative investor who maximizes equation (1) subject to equations (2), (3), (4), (5), and (6) and exhibits CRRA preferences, that is,* $-u'(c)/u''(c) = \gamma c$*, with* γ *the constant that represents relative risk aversion. We normalize the total number of shares and bondholdings to one, i.e.* $n_(0) = m_i(0) = 1$*. The excess return process of risky asset i,* $R_i(t)dt - rdt \equiv \frac{dp_i(t)}{p_i(t)} - rdt$*, that satisfies the equilibrium restriction that* $n_(t) = m_i(t) = 1$ *is given by:*

$$R_i(t)dt - rdt = \gamma\beta_i \left(\frac{W(t)}{p_i(t)}\right) dt + \sigma_i \left(\frac{W(t)}{p_i(t)}\right) dz_i(t), \tag{10}$$

where

$$\beta_i \equiv \sigma_i^2 + \sum_{j\neq i} \rho_{ij}\sigma_i\sigma_j,$$

and $W(t)$ *is total wealth at time t:*

$$W(t) \equiv \sum_{i=1}^{N} p_i(t) + B(t)$$

Proof 1 *See Appendix.*

The theorem implicitly defines the drift and diffusion term that constitute the equilibrium price process, namely $\mu(p_t, t) = r + \beta_i\gamma\frac{W(t)}{p_i(t)}$ and $\sigma(p_t, t) = \sigma_i\frac{W(t)}{p_i(t)}$. Note that the dynamics of the price process do not depend on the investment horizon T of the representative agent, so we can let T approach infinity. The result shows that the magnitude of the risk premium and the associated risk varies over time since the expected return and variance of the return are inversely related to the portfolio weight of the asset, $p_i(t)/W(t)$. This endogenously creates a time-varying risk-premium, betas and volatility that change over time, and many other stylized facts from the empirical literature, which we discuss in the next sections. But before we do so, we also provide the dynamics of the market portfolio of risky assets, or simply market portfolio $V(t)$, which is defined as $V(t) \equiv \sum_{i=1}^{N} p_i(t)$.

Remark 1 *The excess return process of the market portfolio V(t),* $R_V(t)dt - rdt \equiv$

(quadratic utility), Constant Absolute Risk Aversion (CARA), Constant Relative Risk Aversion (CRRA), and Stone-Geary preferences (habits, subsistance levels). However, it turns out that the price process for CARA is very similar and allows us to introduce a non-zero interest rate, so we focus on CARA utility in the main text

$\frac{dV(t)}{V(t)} - rdt$,*that satisfies market equilibrium as defined in Proposition 1 is given by:*

$$R_V(t) - rdt \equiv \frac{dV(t)}{V(t)} - rdt = \gamma\sigma_V^2 \left(1 + \frac{B(t)}{V(t)}\right) dt + \sigma_V \left(1 + \frac{B(t)}{V(t)}\right) dz_V(t), \tag{11}$$

where $\sigma_V^2 = \sum_{i=1}^N \sigma_i^2 + \sum_{i=1}^N \sum_{j\neq i} \rho_{ij}\sigma_i\sigma_j$ *and* $z_V(t) = \sigma_V^{-1} \sum_{i=1}^N z_i(t)$. *Note that* $z_V(t)$ *is standard Brownian motion.*

Again, we observe a time varying risk-premium and time-varying variance, since these depend on the inverse of the deflated price, that is, the price of the market portfolio divided by the price of the bond $V(t)/B(t)$. We now first discuss time-series properties of the market portfolio.

3 Confronting the properties of the market portfolio dynamics with the stylized facts

Stock return distribution

Our main result for the market portfolio dynamics, equation (11), shows that conditionally on the deflated price, returns are normally distributed. But the unconditional returns follow a distribution that is a ratio of normally distributed variables. Distributions of ratios of stochastic variables, such as the Cauchy distribution, are known to exhibit *fat tails*. This means that "bubbles" and "crises" in our return process occur more often than they would under normality. Empirically, fat tails in stock returns is well documented.

Time series properties

Equation (11) shows that conditional expected returns vary over time. The derivative of the drift term with respect to the price is $-(B(t)/V(t)^2)\gamma\sigma_v^2 < 0$: if deflated prices are high, expected returns are low. The intuition is straightforward. After a negative shock, the price of the stock should decline correspondingly. However, investors will also want to move away from risky stock. Since the average investor should hold the market portfolio, the price of the risky stock must decrease even further until the expected return has increased to the point where the average investor is willing to hold the risky stock. Market clearing under fixed asset supply thus generates a time-varying risk premium, and movements in returns that may have been interpreted as "excess volatility".

Indeed, other authors have stressed the importance of incorporating a varying risk-premium in asset pricing models to match the stylized facts of stock returns (e.g. Campbell and Cochrane, 1999), but this variation in the risk premium is usually exogenous to those models. Even though the parameters of the model are fixed over time, our return process exhibits a time-varying risk premium that arises endogenously.

In addition, our return process exhibits other well known time-series properties of stock returns. Since the deflated price does not only predict conditional stock returns, but is also a very persistent time series, returns will exhibit "momentum" (Carhart, 1997) in the short run. At the same time, stock returns cannot stay high for an extended period, since a series of high returns implies a higher deflated price. In time the deflated price will then lower the expected return, so we have "mean reversion" in stock returns in the long run (DeBondt and Thaler, 1985; Poterba and Summers, 1988).

Generalized Auto-Regressive Conditional Heteroskedasticity (GARCH) models and other econometric methods have shown that volatility in stock returns changes over time, and exhibits autocorrelation. Time-varying volatility in these models is taken as exogenous, whereas in our model it also arises endogenously; the derivative of the diffusion term with respect to the price is equal to $-(B(t)/V(t)^2)\sigma_v < 0$. Again, due to both the scaling of the diffusion term by and the persistence of the deflated price, the model can account for periods of high volatility and low volatility. From equation (11) we see that if the deflated price goes down, volatility goes up. In a crisis we thus may expect larger volatility in stock returns. This is the "leverage effect"; a large price decline implies a period of high volatility (Schwert, 1989; Nelson, 1991).

Again, many studies have found variation over time in expected returns. Moreover, several variables that are deflated with the stock price, e.g. dividend-price (D/P) ratios, or earnings-price (E/P) ratios predict expected returns (see e.g. Campbell and Shiller, 1988; Fama and French, 1988; Cochrane, 2008). This is in line with our return process. If D/P or E/P ratios proxy for the deflated price in our return process, equation (11) shows that they should predict conditional expected returns. In fact, Cochrane (2008) disentangles whether D/P ratios predict dividend growth or expected returns, and finds evidence for the latter. The interpretation of Cochrane is similar; low prices signal higher expected returns. The average investor must hold the market portfolio, but is only willing to do so after "bad news" if the expected return goes up.

There is also a link with the habit formation literature. Campbell and Cochrane (1999) model additive habits, which mathematically generates similar expressions to ours - it creates an affine structure in the price dynamics which seems to be an important attribute to explain the stylized facts. However, in their model an additional term in the utility function arises due to habit formation that is independent of and disproportional to consumption. In our case, there is a similar disproportionality, but it arises because of the fact that portfolio weights cannot stay in the same proportion when the number of shares is fixed. One drawback of explaining the stylized facts using habit formation as an economic explanation is that the qualitative features of the return process are sensitive to the way in which habits are introduced. For instance, the results disappear when habit formation is multiplicative (cf. Abel, 1990).

Finally, since in equilibrium prices should be such that the average investor neither saves nor consumes risky stock, consumption growth is not affected by volatility of stock prices. It means that the model is in principle compatible with low and relatively constant interest rates, and roughly i.i.d. consumption growth with small volatility - though admittedly the dynamics of the interest rate is exogenous here and should depend on the assumptions about the supply side of the risk-free asset. In any case, market clearing restrictions under fixed asset supply imply that we should rethink the variables driving the stochastic discount factor. The Hansen-Jagannathan (1991) bounds traditionally imply very high risk-aversion (or a number of other equivalent puzzles), since the stochastic discount factor is based on the marginal utility of consumption. What equation (9) tells us is that, when asset supply is more or less fixed, we should actually use a discount factor that is based on the marginal utility of total wealth.

4 Confronting the crossectional properties of the of asset price dynamics with the stylized facts

The results we discussed in the previous section do not only hold for the market portfolio, but also for individual assets. In this section we highlight some properties that are characteristic for the crossection of asset price returns.

Distribution of asset prices in the market portfolio

Fernholz and Shay (1982) have shown that the distribution of portfolio weights of individual assets in the market portfolio should be stable. Their finding reformulates the fundamental issue we discuss. In short the argument runs as follows. Suppose betas of assets are constant. This implies that the asset with the highest beta should on average have the highest return. Suppose this asset has a beta larger than one. With a fixed number of shares, the growth rate of this asset dominates the other assets and in the long run this asset grows exponentially much faster than the other assets, so it should eventually dominate the portfolio in terms of the portfolio weight. But if the market portfolio is dominated by one large asset, it implies that the beta of this asset should converge to one. This gives a theoretical contradiction, hence returns cannot be i.i.d. Moreover, it also violates the empirical finding by Fernholz et al. (1998) that portfolio weights appear to be stable over time. In light of this, we present an important result regarding the equilibrium distribution of portfolio weights which illustrates how our return process resolves the fundamental contradiction in light of the portfolio weights.

Proposition 2 *Consider the price process as defined by equation (10). The dynamic process of the market portfolio weight of risky asset i , $p_i(t)/V(t)$, is given by:*

$$d(p_i(t)/V(t)) = -\frac{W(t)}{V(t)}\left(\gamma - \frac{W(t)}{V(t)}\right)\left(\sigma_V^2\left(\frac{p_i(t)}{V(t)}\right) - \beta_i\right)dt + \sigma_i\frac{W(t)}{V(t)}dz_i(t) - \left(\frac{W(t)}{V(t)}\frac{p_i(t)}{V(t)}\right)dz_V(t). \quad (12)$$

In addition, this process is mean-reverting if the parameter of risk aversion $\gamma > \frac{W(t)}{V(t)}$. In that case, the long-term equilibrium portfolio weight of asset i is given by $p_i(t)/V(t) = \frac{\beta_i}{\sigma_V^2}$.

Proof 2 *See Appendix.*

The proposition basically shows that portfolio weights converge to their "fundamental betas ", $\frac{\beta_i}{\sigma_V^2}$, i.e. the correlation of the Brownian motion of the individual assets with the Brownian motion of the market portfolio. The reason is intuitive, equation (10) shows that the drift term of individual stock returns is inversely related to the portfolio weight. As the price of the asset increases, its return will go down relative to the other assets. Cochrane (2008) finds similar effects in his two-tree model.

Time-varying betas and the value and size effect.

The price process given by equation (10) can be rewritten in a more familiar form, so that the excess returns of individual assets are related to the excess returns of the market portfolio:

$$R_i(t)dt - rdt = \frac{\beta_i}{\sigma_V^2}\frac{V(t)}{p_i(t)}\left(R_V(t)dt - rdt\right) + \left(\frac{W(t)}{p_i(t)}\right)\left(\sigma_i dz_i(t) - \frac{\beta_i}{\sigma_V}dz_V(t)\right).$$

We observe that the conventional beta, i.e. the coefficient in front of the excess return of the market portfolio, varies over time and is given by $\frac{\beta_i}{\sigma_V^2}\frac{V(t)}{p_i(t)}$.

Stocks thus have periods of low and high betas, depending on their relative market value, i.e. their portfolio weight. However, from proposition 2 we know that the portfolio weight $\frac{p_i(t)}{V(t)}$ will converge to a value of $\frac{\beta_i}{\sigma_V^2}$, implying that over time the conventional betas will converge to a value of one. This mean reversion in betas is well-documented (see e.g. Blume, 1975) , moreover it is known that over longer time periods betas are on average equal to one (see e.g. Keim, 1983). The implication of our results is that when one sorts portfolios on betas and calculates the average returns for the next twelve months, the betas and the associated returns for these next twelve months will have converged to their equilibrium values, yielding a security market line that is too flat (cf. Fama and French, 2004). These observations provide interesting insights, summarized in the next remark.

Remark 2 *The conditional "beta", $b_i(t)$, defined as the conditional correlation of the excess return $R_i(t) - rdt$ of asset i with the excess return of the market portfolio of risky assets, $R_V(t) - rdt$, and the conditional expected excess return of asset i, $E_t[R_i(t) - rdt]$, are given by:*

$$b_i(t) = \frac{\beta_i}{\sigma_V^2} \frac{V(t)}{p_i(t)}, \tag{13}$$

$$E_t[R_i(t) - rdt] = b_i(t)[R_V(t) - r]dt = b_i(t)\gamma\sigma_V^2 \frac{W(t)}{V(t)} dt. \tag{14}$$

We observe that, conditionally, betas and expected returns should be on a straight line. However, the slope of this line varies over time, depending on the current market risk-premium, which depends on the current leverage in the market $\gamma\sigma_V^2 \frac{W(t)}{V(t)}$. As said, betas also vary over time, and are related to the inverse of the portfolio weight of the asset. If the price of an asset is relatively low, i.e. the portfolio weight is lower compared to the "equilibrium" portfolio weight discussed in proposition 2, the beta of this asset will be high, but the expected return can be disproportionally high, depending on the current slope of the beta-expected return line. This explains why we might observe a "value" or "size" effect, cf. Fama and French (1993). However, it depends on the historical evolution of the "beta slope" whether or not we observe these effects. This may explain why the size effect disappeared since the 1980s, while the value effect was present in U.S. stocks between 1963-2009, but not between 1926–1963 (see Cochrane, 2011). Note that both the time-varying betas and the time-varying trade-off between expected returns and beta arise endogenously, since all the fundamental parameters of the model are assumed to be fixed over time.

So the upshot is that if we want to describe the beta-return relationship we need to account for both time-varying betas, as well as, a varying risk premium, as is also pointed out by Cochrane (2011). Our general equilibrium model provides us with guidelines on how one could proceed. Figure 2 gives a summary of all our results and an overview of how the security market line dynamically behaves according to our model.

Figure 2 shows that the slope of the security market line is varying over time. In bad times, when prices are low, expected returns are relatively higher, for each value of the conditional beta $b_i(t)$. The slope will always converge back to some equilibrium value, which we labeled "Normal Times". Not only the market portfolio has a beta equal to one, but in principal all risky assets have an equilibrium value for beta equal to one. Stocks that have relatively low values (low portfolio weight), will have a higher beta, and hence a higher return. However in the long run, they mean revert to a beta equal to one.

As mentioned, studies that plot betas of buy and hold portfolios actually find a large cluster around beta equal to one (see e.g. Keim, 1983), with usually similar expected returns. This is in line with what our model predicts. However, in order to explain differences in asset returns, it may be better to do yearly sorts of portfolios to make sure one observes sufficient variation in both conditional betas and average returns, in line with, for example, the Fama and French (1993) sorts. Note that it is not surprising that Fama and French need to construct the portfolios on a yearly basis, as in the long-run the betas and returns will converge, leaving little crossectional variation to explain. The figure shows that sorting in high and low market-to-book values is actually a good idea, as this will very likely generate variation in the conditional betas.

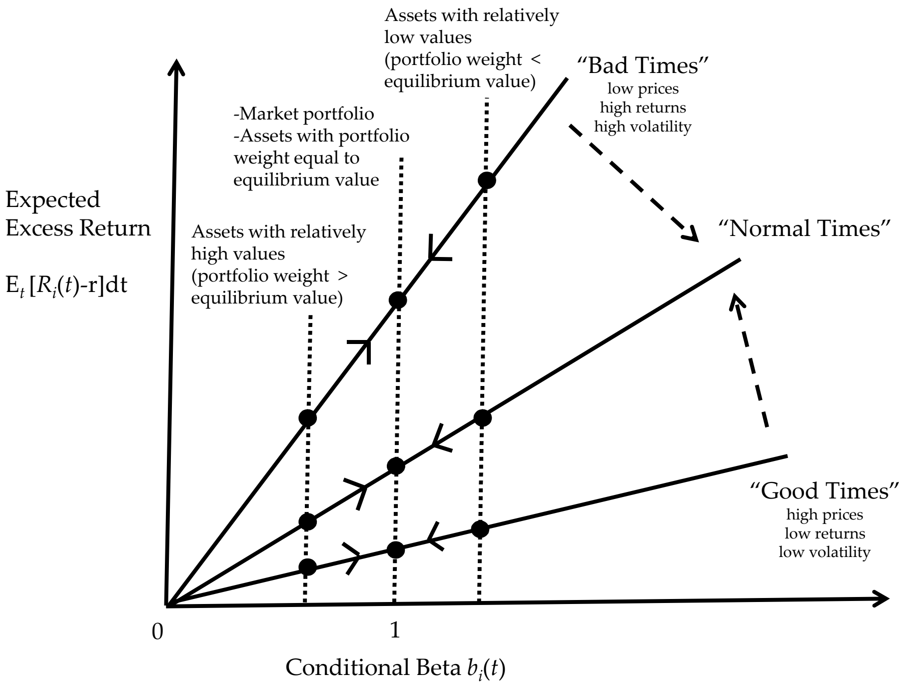

Figure 2. Dynamic behavior of the security market line

Notes: The main equation for the return process is given by $R_i(t)dt - rdt = \gamma\beta_i \left(\frac{W(t)}{p_i(t)}\right) dt + \sigma_i \left(\frac{W(t)}{p_i(t)}\right) dz_i(t)$. For this figure, we rewrite the return process of the individual assets in terms of the return process of the market portfolio of risky assets, which turns out to be linear, though the coefficient is time-varying, and given by: $E_t[R_i(t) - rdt] = b_i(t)[R_V(t) - r]dt$, with the conditional beta defined as: $b_i(t) = \frac{\beta_i}{\sigma_V^2}\frac{V(t)}{p_i(t)}$, and the expected excess return of the market portfolio given by $E_t[R_V(t) - r]dt = \gamma\sigma_V^2 \left(\frac{W(t)}{V(t)}\right) dt$. The fact that the excess return on the market portfolio is time-varying makes the slope of the security market line time-varying. In addition, the conditional betas depend on the inverse of the portfolio weight, so stocks with relatively low market values will have high betas. Moreover, the betas are mean-reverting to a value of one, driving the relative prices back to the equilibrium portfolio weight.

5 Conclusion

In this paper, we argue that general equilibrium restrictions need to play a more central role in asset pricing models. We present a simple dynamic asset pricing model that is able to explain many stylized facts from the finance literature. The main innovation is that we assume fixed asset supply (a fixed number of shares). Using a standard framework, we solve for equilibrium prices and characterize the resulting price process. We confront the results with the stylized facts.

The appeal of our model is its simplicity. The model contains only a few time-invariant parameters and yet is able to endogenously generate features such as a varying risk-premium, varying betas, and volatility clustering. We show how the conventional security market line should incorporate changes in the risk-premium and time-varying betas. Empirically the model is also attractive, since the equations for the return processes are intuitive and parsimonious.

The model has a number of limitations. First of all, we limit ourselves to HARA utility, with the main focus on CARA. The expressions in most other cases are tedious and less intuitive though, so for sake of exposition, we choose for a tractable and simple model. Also, the price of the risk-free asset is exogenous. One would need to include the real-side of the economy with productive capital in the model to deduce an endogenous interest rate. Finally, the process we discuss can only be time-homogeneous if one allows the supply side of the risk free asset to have a growth rate equal to the risk premium - a requirement which is in line with the stylized facts. If not, the deflated prices we discuss will explode in the long run. At this stage, however, we have not been able to find closed form solutions for such an economy, and leave this to future research.

Then again, the main purpose of this paper is not to solve every complication of the model. Our main point is simply that general equilibrium and the fact that the number of shares is empirically more or less constant is not compatible with i.i.d returns and can explain many empirical stylized facts. Others have mentioned this before, but we hope to have convinced the reader that equilibrium restrictions may be more important for empirical asset pricing than has been considered so far.

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Mathematical Appendix

Derivation of the first-order conditions of the investors maximization problem

Solving the maximization problem with backward induction, we find the following first-order condition at $t = T - dt$:

$$-u'(c(T-dt))p_i(T-dt) + e^{-\rho dt}E_{T-dt}[u'(c(T))p_i(T)] = 0,$$

which is an implicit demand function for the number of shareholdings $m_i(t)$ at $t = T - dt$. Prices should equate supply with demand, so $m_i(t) = m_i(0)$ implies equilibrium. We thus find a standard expression for the price of the risky stock at $t = T - dt$:

$$p_i(T-dt) = e^{-\rho dt}\frac{E_{T-dt}[u'(c(T))p_i(T)]}{u'(c(T-dt))},$$

Once $p_i(T - dt)$ is known we can find the first order condition at $t = T - 2dt$:

$$-u'(c_(T-2dt))p_i(T-2dt) + e^{-2\rho dt}E_{T-2dt}[u'(c(T-dt))p_i(T-dt)] = 0,$$

We can substitute the expression for p_{T-dt} in this equation, again impose equilibrium, and find:

$$-u'(c_(T-2dt))p_(T-2dt) + e^{-2\rho dt}E_{T-2dt}[E_{T-dt}[u'(c(T))p_i(T)]] = 0.$$

Iteratively solving the first-order conditions, we find an expression for the equilibrium price at time t:

$$p_i(t) = e^{-\rho(T-t)}\frac{E_t[u'(c(T))p_i(T)]}{u'(c(t))}.$$

The first order condition for the risk-free asset is derived similarly:

$$-u'(c(T-dt))B(t) + e^{(r-\rho)dt}E_{T-dt}[u'(c(T))]B(T) = 0.$$

This yields:

$$B(t) = B(T)e^{-\rho(T-t)}\frac{E_t[u'(c(T))]}{u'(c_t)}$$

Which are the two first-order conditions in the text.

Proof of proposition 1

To show that the price process is indeed an equilibrium price process we solve the stochastic differential equation given by (10). To do so, we first solve the stochastic differential equation for the wealth, which reads:

$$\frac{dW(t)}{W(t)} = rdt + \gamma\sigma_V^2 dt + \sigma_V dz_V(t),$$

since $dW(t) = d[\sum_{i=1}^N p_i(t) + B(t)] = \sum_{i=1}^N dp_i(t) + dB(t)$. Using Itō's lemma, we find for $w(t) \equiv W(t)/B(t)$:

$$\frac{dw(t)}{w(t)} = \gamma\sigma_V^2 dt + \sigma_V dz_V(t)$$

hence

$$w(t) = w(0)e^{(\gamma - \frac{1}{2})\sigma_V^2 t + \sigma_V z_V(t)},$$

So we find for the wealth at time t:

$$W(t) = B(t)W(0))e^{(\gamma - \frac{1}{2})\sigma_V^2 t + \sigma_V z_V(t)}$$

We can rewrite the stochastic differential equation of the individual assets:

$$d(p_i(t)/B(t)) = \frac{\beta_i}{\sigma_V^2} dw(t) - \frac{\beta_i}{\sigma_V} w(t) dz_V(t) + \sigma_i w(t) dz_i(t),$$

so

$$p_i(t)/B(t) - p_i(0)/B(0) = \int_0^t \frac{\beta_i}{\sigma_V^2} dw(s) - \frac{\beta_i}{\sigma_V} w(s) dz_V(s) + \sigma_i w(s) dz_i(s)$$

$$= \frac{\beta_i}{\sigma_V^2}(w(t) - w(0)) + \int_0^t -\frac{\beta_i}{\sigma_V} w(s) dz_V(s) + \sigma_i w(s) dz_i(s)$$

Hence, we find the solution for the stochastic differential equation for asset i:

$$p_i(t) = B(t)\left(p_i(0)/B(0) + \frac{\beta_i}{\sigma_V^2}(w(t) - w(0)) + \int_0^t -\frac{\beta_i}{\sigma_V} w(s) dz_V(s) + \sigma_i w(s) dz_i(s)\right)$$

Next we check whether this price is indeed an equilibrium price. The pricing equation of an individual asset reads:

$$p_i(t) = \frac{B(t)}{B(T)} \frac{E_t[u'(c(T))p_i(T)]}{E_t[u'(c(T))]} \tag{15}$$

We need to evaluate the right-hand side of the pricing equation to verify whether the price process is indeed an equilibrium process;

$$\frac{B(t)}{B(T)} \frac{E_t[u'(c(T))p_i(T)]}{E_t[u'(c(T))]} = \frac{B(t)}{B(T)} \frac{E_t[W(T)^{-\gamma} p_i(T)]}{E_t[W(T)^{-\gamma}]}$$

$$= \frac{B(t)}{B(T)} \frac{E_t[w(T)^{-\gamma} p_i(T)]}{E_t[w(T)^{-\gamma}]}$$

$$= B(t) \frac{E_t[(w(T)^{-\gamma} \left(p_i(0)/B(0) + \frac{\beta_i}{\sigma_V^2}(w(T) - w(0)) + \int_0^T -\frac{\beta_i}{\sigma_V} w(s) dz_V(s) + \sigma_i w(s) dz_i(s) \right)]}{E_t[(w(T)^{-\gamma}]}$$

$$= B(t) \left(\frac{p_i(0)}{B(0)} - \frac{\beta_i}{\sigma_V^2} w(0) + \frac{E_t[(w(T)^{-\gamma} \left(\frac{\beta_i}{\sigma_V^2} w(T) - \int_0^T \frac{\beta_i}{\sigma_V} w(s) dz_V(s) - \sigma_i w(s) dz_i(s) \right)]}{E_t[(w(T)^{-\gamma}]} \right)$$

$$= p_i(t) + \frac{B(t) E_t[(w(T)^{-\gamma} \left(- \int_t^T \frac{\beta_i}{\sigma_V^2} w(s) dz_V(s) - \sigma_i w(s) dz_i(s) \right)]}{E_t[(w(T)^{-\gamma}]}$$

$$= p_i(t) +$$

$$\frac{B(t) E_t[\left(\int_0^T w(s)^{-\gamma} (\frac{1}{2}(\gamma - \gamma^2) \sigma_V^2 ds - \gamma \sigma_V dz_V(s)) \right) \left(- \int_t^T \frac{\beta_i}{\sigma_V} w(s) dz_V(s) - \sigma_i w(s) dz_i(s) \right)]}{E_t[(w(T)^{-\gamma}]}$$

$$= p_i(t) + \frac{B(t) E_t[\left(\int_t^T w(s)^{-\gamma} \gamma \sigma_V dz_V(s) \right) \left(\int_t^T \frac{\beta_i}{\sigma_V} w(s) dz_V(s) - \sigma_i w(s) dz_i(s) \right)]}{E_t[(w(T)^{-\gamma}]}$$

$$= p_i(t) + \frac{B(t) \gamma \beta_i \left(E_t[\int_t^T w(s)^{1-\gamma} ds] - E_t[\int_t^T w(s)^{1-\gamma} ds] \right)}{E_t[(w(T)^{-\gamma}]}$$

$$= p_i(t)$$

Price process for HARA utility and zero interest rate

Proposition 3 *Consider a representative investor who maximizes equation (1) subject to equations (2), (3), (4), (5), and (6) and exhibits HARA preferences, that is, $-u'(c)/u''(c) = \gamma(c + \kappa)$, with γ, κ constant. We assume $r = 0$ and normalize the total number of shares and bondholdings to one, i.e. $n_i(0) = m(0) = 1$.*

The return process of risky asset i, $R_i(t)$, that satisfies the equilibrium restriction that $n_i(t) = m(t) = 1$ is given by:

$$R_i(t)dt \equiv \frac{dp_i(t)}{p_i(t)} = \beta_i \gamma \left(\frac{W(t) + \kappa}{p_i(t)} \right) dt + \sigma_i \left(\frac{W(t) + \kappa}{p_i(t)} \right) dz_i(t), \qquad (16)$$

where

$$\beta_i \equiv \sigma_i^2 + \sum_{j \neq i} \rho_{ij} \sigma_i \sigma_j,$$

and $W(t)$ is total wealth at time t:

$$W(t) \equiv \sum_{i=1}^{N} p_i(t) + B(t)$$

Proof 3 *The proof goes analogues to the proof of proposition 1.*